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TRANSFINITE DESCENDING SEQUENCES OF MODELS HOD$^\alpha$

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0. Introduction

There are some models of set theory properties of which seem to be particularly worth studying: L, $L[U]$ (if a measurable cardinal exists), K (the core model), $L[R]$ (if we assume the axiom of determinacy), finally HOD (the class of hereditarily ordinal definable sets). All of them are inner models of set theory i.e. all the axioms of ZF hold when relativized to any of them, provided they hold in the universe.

The main difference between the last one and the others lies in the non-absoluteness of HOD: when one performs the construction of HOD inside the HOD already constructed it may happen that the resulting model HOD^2 will be smaller.

We will return to the problems of the iteration of HOD after general remarks concerning ordinal definability.

A set is said to be *definable* if there is a formula of set theory which defines it (no parameters are allowed). The property of being definable need not to be expressible within ZF set theory. If it were, it would lead to the paradox of the first non-definable ordinal (assuming that there are uncountable many of them). But the property of being ordinal definable is expressible by a formula of set theory. Also there exist a formula of set theory expressing the property of a set of being *hereditarily* (with respect to the $\in$-relation) *ordinal definable* (cf. Section 1.1). This fact is a consequence of the reflection principle.

The notion of ordinal definability was introduced by Gödel. Myhill and Scott [7] described elementary properties of the class HOD, in particular, their paper contains a proof of the fact that the axiom of choice is always true in HOD. McAloon [6] showed that it is consistent that HOD $\neq L$. Roguski [9] developed his method and showed that any model of ZFC can be HOD of a certain generic extension, and that any model of ZFC can be extended to a model satisfying $V = \text{HOD}$. In conclusion he obtained that $\text{ZF} \vdash (\phi)^{\text{HOD}}$ iff $\text{ZFC} \vdash \phi$, for any formula ϕ of set theory.

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Since HOD is not absolute, it can be iterated: $\mathrm{HOD}^{\alpha+1} = (\mathrm{HOD})^{\mathrm{HOD}^\alpha}$, $\mathrm{HOD}^\lambda = \bigcap_{\alpha<\lambda} \mathrm{HOD}^\alpha$ for limit λ. Clearly, HOD^n, $n<\omega$, is a model of ZFC. But HOD^ω may fail to satisfy the axiom of choice (McAloon), or even fail to satisfy ZF (L. Harrington used a proper class of conditions to obtain this result). On the other hand, Grigorieff [3] proved that if the universe is constructible from a set, then HOD^α's form a definable sequence of models of ZF, it is however constant for all α's greater than some fixed ordinal. Thus it is impossible to obtain a model of ZF in which the sequence HOD^α, $\alpha<\mathrm{On}$, is strictly descending, via forcing with a set of conditions. In this paper we prove some general facts about the structure of iterated HOD when one forces with some Easton-like class of conditions. We show that then the relation $x \in \mathrm{HOD}^\alpha$ is expressible by a formula of set theory, and so HOD^α's, $\alpha<\mathrm{On}$, are models of ZF. As a corollary to the general theorems we obtain the main result: *It is consistent* (*with* ZFC) *that there is a strictly descending sequence of models* HOD^α, $\alpha<\mathrm{On}$.

The theorem is proved by constructing a model of ZF set theory in which it holds. The model is a generic extension of L by a class of conditions which is a product of Souslin trees constructed by Jech [4]. Jech used these trees to show the consistency of existence of the iterations of length of any cardinal number. The paper uses many of Grigorieff's results about intermediate models of set theory [3]. We want to mention here that the result can be proved by the McAloon–Roguski type of forcing.

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1. Preliminaries on forcing

1.0. Notation

Standard notation is used throughout.

For any set A, the cardinality of A is denoted by $\bar{\bar{A}}$. The letters b.a., c.b.a., c.b.s. denote respectively: 'Boolean algebra', 'complete Boolean algebra', 'complete Boolean subalgebra'. κ, κ_0, κ_1, $\kappa_2, \ldots$ always denote cardinals. λ, λ' are reserved for limit ordinals. c.t.m. means 'countable transitive model'. $\mathcal{M}$, $\mathcal{N}$ are reserved for c.t.m. of ZF. $(\ )^N$ means that the notion in the parentheses is relativized to the structure N. If it is clear from the context in which structure $(\ )$ is defined, we will drop the superscript.

1.1. Basic definitions and theorems (cf. [3])

Definition 1.1. (i) $\mathrm{ZF}(U) = \mathrm{ZF}+$'all instances of the replacement scheme for formulae involving the predicate U'.

(*ii*) X is a *class* for (in) $\mathcal{N}$ iff $(\mathcal{N}, X) \models \mathrm{ZF}(U)$.

Definition 1.2. (i) a is *definable* in $\mathcal{N}$ from (elements of) Y iff there is a formula ϕ and finitely many parameters $p_1, p_2, \ldots, p_n \in Y$ s.t. $\mathcal{N} \models \forall x(x \in a \equiv \phi(x, p_1, \ldots, p_n))$.

(ii) a is *ordinal definable* in $\mathcal{N}$ iff a is definable in $\mathcal{N}$ from elements of On.

(iii) $\mathrm{OD} = \{a\colon a \text{ is ordinal definable}\}$,
$\mathrm{HOD} = \{a\colon \mathrm{TC}(\{a\}) \subseteq \mathrm{OD}\}$.

Definition 1.3. (i) $L[x]$ denotes the smallest model of ZF which contains x (as a subset).

(ii) $L[X] = \bigcup \{L[V_\alpha \cap X]\colon \alpha < \mathrm{On}\}$, in details:

$$L_{\beta+1}[X] = \mathrm{Def}\langle L_\beta[X], \varepsilon, X \cap L_\beta[X]\rangle \cup V_{\beta+1} \cap \mathrm{TC}(X),$$
$$L_\lambda[X] = \bigcup \{L_\beta[X]\colon \beta < \lambda\}, \quad \lambda \leqslant \mathrm{On}.$$

Proposition 1.1. (i) *If X is a class for $\mathcal{N}$, then $L[X]$ is a model of* ZF.

(ii) *If $X \subseteq \mathrm{On}$, then $L[X]$ satisfies* AC.

(iii) *Let* rank $y_0 = \tau$, *then* $y_0 \in L_{\tau+1}[y_0]$.

Definition 1.4. If $\mathcal{M}$ is a class for $\mathcal{N}$, define

$$\mathcal{M}[x] = L[\mathcal{M} \cup \{x\}], \quad (x \in \mathcal{N}).$$

Proposition 1.2 (The product lemma). *Let $\langle \mathbb{P}_1, \leqslant_1\rangle$, $\langle \mathbb{P}_2, \leqslant_2\rangle$ be notions of forcing belonging to $\mathcal{M}$. Let $\mathbb{P}$ be their product. Then, for any $\mathbb{P}$-generic over $\mathcal{M}$ set G, there are G_1. G_2 s.t.*

(i) $G = G_1 \times G_2$,

(ii) *G_1 is $\mathbb{P}_1$-generic over $\mathcal{M}$, and G_2 is $\mathbb{P}_2$-generic over $\mathcal{M}[G_1]$.*

(iii) $\mathcal{M}[G] = \mathcal{M}[G_1][G_2]$.

Moreover, if H_1 is $\mathbb{P}_1$-generic over $\mathcal{M}$ and H_2 is $\mathbb{P}_2$-generic over $\mathcal{M}[H_1]$, then $H_1 \times H_2$ is $\mathbb{P}$-generic over $\mathcal{M}$.

Proposition 1.3. *If $\mathcal{M}$ is an inner model of $\mathcal{N}$, C is a notion of forcing in $\mathcal{M}$, and G is C-generic over $\mathcal{N}$, then $\mathcal{M}[G] = \mathcal{N}[G]$ implies $\mathcal{M} = \mathcal{N}$.*

Theorem 1.4 (Vopênka–Balcar [10]). *Let $\mathcal{M}$, $\mathcal{N}$ be transitive models for* ZF *with the same sets of ordinals. If one of them satisfies* AC, *then $\mathcal{M} = \mathcal{N}$.*

Definition 5.1. (i) A notion of forcing $\mathbb{P}$ satisfies *κ-descending chain condition* (κ-d.c.c.) or is κ-closed iff for any descending sequence of conditions in $\mathbb{P}$, $p_0 \geqslant p_1 \geqslant \cdots \geqslant p_\xi \geqslant \cdots \xi < \kappa$, there is $p \leqslant p_\xi$ for all $\xi < \kappa$.

(ii) $\mathbb{P}$ satisfies the *κ-chain condition* (κ-c.c.), if every set of pairwise incompatible conditions has power less than κ.

(iii) If B is a c.b.a. in $\mathcal{M}$, we say that B is (κ, ∞)-*distributive* iff for any G B-generic over $\mathcal{M}$, $({}^\kappa\mathcal{M})^{\mathcal{M}[G]} = ({}^\kappa\mathcal{M})^{\mathcal{M}}$.

(iv) We say that algebra B is κ-closed, if there is $P \subseteq B$ s.t. P is dense in B and P is κ-closed.

1.2. Forcing with proper classes

Assume $C = \langle C, \leqslant \rangle$ is a notion of forcing in $\mathcal{M}$. C is a class for $\mathcal{M}$. Let $\langle \kappa_\alpha : \alpha \in \mathrm{On} \rangle$ be an ascending sequence of cardinals which is definable in $\mathcal{M}$. Assume also that for each $\alpha \in \mathrm{On}$ there is a definable decomposition of $C \simeq C_\alpha \times C^\alpha$, where C_α satisfies κ_α^+-c.c., and C^α satisfies κ_α-d.c.c.

Theorem 1.5 (Easton [2]). *Assume that the conditions above are satisfied. Then $\mathcal{M}[G]$ is a model for* ZF, *for any G which is C-generic over $\mathcal{M}$. If $\mathcal{M} \vDash \mathrm{AC}$, then $\mathcal{M}[G] \vDash \mathrm{AC}$.*

Note. From now on, classes of conditions which will appear in this dissertation will satisfy these Easton's conditions.

Proposition 1.6 (The product lemma for classes). *Let C be a class of conditions in $\mathcal{M}$, let G be C-generic over $\mathcal{M}$. Then*

(i) *$\mathcal{M}[G] = \mathcal{M}[G \cap C_\alpha][G \cap C^\alpha]$ and $G \cap C_\alpha$ is C_α-generic over $\mathcal{M}$ and $G \cap C^\alpha$ is C^α-generic over $\mathcal{M}[G \cap C_\alpha]$.*

(ii) *$\mathcal{M}[G] = \mathcal{M}[G \cap C^\alpha][G \cap C_\alpha]$ and $G \cap C_\alpha$ is C_α-generic over $\mathcal{M}[G \cap C^\alpha]$.*

For the proof of this proposition the reader can consult Zarach [11].

Proposition 1.7 (Jech [4]). *Let A be a (κ, ∞)-distributive c.b.a. Let $\mathrm{A}_0, \mathrm{A}_1, \ldots, \mathrm{A}_\alpha$, $\alpha < \kappa$, be a κ-sequence of c.b.a.s s.t. $\mathrm{A}_0 = \mathrm{A}$, $\mathrm{A}_{\alpha+1}$ is a c.b.s. of A_α, and $\mathrm{A}_\lambda = \bigcap_{\alpha<\lambda} \mathrm{A}_\alpha$. Let all this be in a ground model $\mathcal{M}$ of* ZFC, *let G be A-generic over $\mathcal{M}$, let $G_\alpha = G \cap \mathrm{A}_\alpha$ for all $\alpha \leqslant \kappa$.*

If x is a set of ordinals and $x \in \mathcal{M}[G_\alpha]$ for all $\alpha < \kappa$, then $x \in \mathcal{M}[G_\kappa]$.

Corollary 1.8 (Jech [4], p. 406).

$$\bigcap_{\alpha<\kappa} \mathcal{M}[G_\alpha] = \mathcal{M}[G_\kappa].$$

By Proposition 1.7 we obtain:

Corollary 1.9. *Let A_α, $\alpha < \kappa$, be a sequence of definable in $\mathcal{M}$ classes of conditions which are (κ, ∞)-distributive, i.e. $\mathcal{M} \vDash b \in \mathrm{A}_\alpha \equiv \phi(b, p, \alpha)$ for some formula ϕ and $p \in \mathcal{M}$. Assume also that we have an ascending sequence of cardinals (in $\mathcal{M}$) $\langle \kappa_\eta : \eta < \mathrm{On} \rangle$ s.t. for each η, each A_α is decomposable into $(\mathrm{A}_\alpha)_\eta$ and $(\mathrm{A}_\alpha)^\eta$ s.t. $\mathrm{A}_\alpha \simeq (\mathrm{A}_\alpha)_\eta \times (\mathrm{A}_\alpha)^\eta$, where $(\mathrm{A}_\alpha)^\eta$ is κ_η-closed, and $(\mathrm{A}_\alpha)_\eta$ satisfies κ_η^+-c.c. Assume also that $(\mathrm{A}_{\alpha+1})_\eta$ is a c.b.s. of $(\mathrm{A}_\alpha)_\eta$, $(\mathrm{A}_\lambda)_\eta = \bigcap_{\alpha<\lambda} (\mathrm{A}_\alpha)_\eta$, and $\mathrm{A}_{\alpha+1}$ is a c.b.s. of A_α, and $\mathrm{A}_\lambda = \bigcap_{\alpha<\lambda} \mathrm{A}_\alpha$, $\lambda \leqslant \kappa$.*

If G is A_0-generic over $\mathcal{M}$, then

$$(1)\qquad (\forall X \subseteq \mathrm{On})\ \Big(X \in \bigcap_{\alpha<\kappa} \mathcal{M}[G_\alpha] \Rightarrow X \in \mathcal{M}[G_\kappa]\Big),$$

$$(2)\qquad \bigcap_{\alpha<\kappa} \mathcal{M}[G_\alpha] = \mathcal{M}[G_\kappa], \quad \textit{where} \quad G_\alpha \stackrel{\mathrm{df}}{=} G \cap A_\alpha.$$

Proof. By Theorem 1.4 it suffices to take care of subsets of On. So, let $X \subseteq \rho \in \mathrm{On}$, $X \in \bigcap_{\alpha<\kappa} \mathcal{M}[G_\alpha]$. Then there is η s.t. for each α, $X \in \mathcal{M}[G \cap (A_\alpha)_\eta]$, and $\kappa_\eta > \rho$.

So $X \in \bigcap_{\alpha<\kappa} \mathcal{M}[G \cap (A_\alpha)_\eta] = \mathcal{M}[G \cap (A_\kappa)_\eta]$, (by Corollary 1.8). Thus $X \in \mathcal{M}[G \cap A_\kappa] = \mathcal{M}[G_\kappa]$.

Proposition 1.10 (The Easton's lemma). *If $\mathbb{P}_1, \mathbb{P}_2$ are notions of forcing in $\mathcal{M}$ s.t. $\mathbb{P}_1$ satisfies κ^+-c.c. and $\mathbb{P}_2$ is κ-closed, then, for any $G = G_1 \times G_2$ which is $\mathbb{P}_1 \times \mathbb{P}_2$-generic over $\mathcal{M}$, if $f \in \mathcal{M}[G]$ is a function from κ into $\mathcal{M}$, then $f \in \mathcal{M}[G_1]$.*

For a proof the reader can consult [5].

2. The model

In this chapter we will define the model in which HOD can be iterated On-many times and the sequence of models HOD$^{\alpha}$, $\alpha < \mathrm{On}$, is strictly descending. However the proof of these two facts will require some work. This will be done in the next chapters.

2.0

Definition 2.1. *Let* $\Gamma_0 = \omega_0$ and $\Gamma_\alpha = \omega_{\Gamma'_\alpha + 25}$, where $\Gamma'_\alpha = (\sup_{\beta<\alpha} \Gamma_\beta)^+$.

Definition 2.2. Let $T^0_\alpha \in L$ be the special Souslin tree on Γ^+_α constructed by Jech in [4]. Let T^γ_α be the appropriate projection of T^0_α defined in the same paper. If b^0_α is a generic over L branch through T^0_α, then b^γ_α denotes its projection on T^γ_α i.e. b^γ_α is the generic over L branch through T^γ_α.

Theorem 2.1 (Jech [4]). *Under the assumptions of Definition 2.2, the corresponding (to T^γ_α's) c.b.a.'s. B^γ_α, $\gamma < \Gamma_\alpha$, form a descending sequence, and $B^\lambda_\alpha = \bigcap_{\gamma<\lambda} B^\gamma_\alpha$ $(\lambda < \Gamma_\alpha)$. Moreover $L[b^0_\alpha] \models \mathrm{HOD}^\gamma = L[b^\gamma_\alpha]$, for all $\gamma < \Gamma_\alpha$, and $\bigcap_{\gamma<\Gamma_\alpha} L[b^\gamma_\alpha] = L$.*

2.1

We have then a family of trees T^0_α, each tree provides the iteration for Γ_α-many steps, so it is natural to suppose that the whole family will give the iteration of length On.

From now on M is a c.t.m. of ZF + $V = L$.

Definition 2.3 (The class of conditions).

$$C=\prod_{\alpha<\mathrm{On}} T^0_\alpha=\{f: (\exists\gamma)(\operatorname{dom} f\subseteq\gamma\ \&\ (\forall\alpha)(\alpha\in\operatorname{dom} f\Rightarrow f(\alpha)\in T^0_\alpha))\}.$$

$$C_\alpha=\{f: f\in C\ \&\ \operatorname{dom} f\subseteq\alpha\},$$

$$C^\alpha=\{f\in C: \operatorname{dom} f\cap\alpha+1=\emptyset\},$$

$$C(\check{\alpha})=\{f\in C: \operatorname{dom} f\cap\{\alpha\}=\emptyset\},$$

$f\leqslant g$ iff $\operatorname{dom} f\supseteq\operatorname{dom} g$ and $(\forall\alpha)(\alpha\in\operatorname{dom} g\Rightarrow f(\alpha)\leqslant_\alpha g(\alpha))$ ($\leqslant_\alpha$ is the ordering of T^0_α).

Proposition 2.2. *If G is C-generic over M, then $M[G]$ is a model of* ZFC.

Proof. $\bar{\bar{C}}_\alpha<\Gamma_\alpha$, i.e. we have Γ_α-c.c. for C_α, so it suffices to show the following.

Claim. C^α satisfies Γ_α-d.c.c.

Proof of the claim. Note that, if each P_ξ satisfies κ-d.c.c., then so does $\prod_{\xi<\rho} P_\xi$ (for any ($\rho\leqslant\mathrm{On}$). So it suffices to show that each $T^0_{\alpha'}$, for $\alpha'>\alpha$, satisfies Γ_α-d.c.c. This is true since [4, p. 406], if T were the special Souslin tree on κ^{++} constructed by Jech, then on the stages (of construction) of cofinality less than κ^+ all the branches were extended. $T^0_{\alpha+1}$ is the tree on $\omega_{\Gamma'_{\alpha+1}+25}$, and $\Gamma'_{\alpha+1}>\Gamma_\alpha$.

The proposition follows then by Theorem 1.5 applied to $(C_\alpha\times T^0_\alpha)\times C^\alpha$.

Proposition 2.3. *If G is C-generic over M, then $G=\prod_{\delta<\mathrm{On}} b^0_\delta$, and each b^0_δ is T^0_δ-generic over $M[\prod_{\delta'\neq\delta} b^0_{\delta'}]$. ($b^0_\delta$ is defined in natural way as $\{b\in T^0_\delta: (\exists f\in G)\times(f(\delta)=b)\}$.)*

Proof. Since G is an ultrafilter, $G=\prod_{\delta<\mathrm{On}} b^0_\delta$. C is in a natural way isomorphic to $T^0_\alpha\times C(\check{\alpha})$, hence the second part of the proposition follows by Proposition 1.6 applied to $C_\alpha\times T^0_\alpha\times C^\alpha\simeq C$.

3. Ordinal definability I

This part contains some important definitions and basic forcing results which will be necessary later. We need also to prove some general results about models HOD X. This will be done by reformulating or strengthening the theorems obtained by Grigorieff [3].

Note. The symbol $[m.n]$ denotes a reference to Section $m.n$ in [3].

3.0

Definition 3.1. Let $X\subseteq\mathcal{N}$, $\mathcal{N}\models$ZF. We define

(i) OD $X=\{a$: a is definable (in $\mathcal{N}$) from ordinals and elements of $X\}$,

(ii) HOD $X=\{a: \mathrm{TC}(\{a\})\subseteq \mathrm{OD}\ X\}$.

Definition 3.2. Let $\mathcal{M}$ be an inner model of $\mathcal{N}$, $x \in \mathcal{N}$.

$$\mathrm{OD}\,\mathcal{M}[x] = \mathrm{OD}(\mathcal{M} \cup \mathrm{TC}(\{x\})),$$
$$\mathrm{HOD}\,\mathcal{M}[x] = \mathrm{HOD}(\mathcal{M} \cup \mathrm{TC}(\{x\})),$$
$$\mathrm{OD}\,\mathcal{M}x = \mathrm{OD}(\mathcal{M} \cup x),$$
$$\mathrm{HOD}\,\mathcal{M}x = \mathrm{HOD}(\mathcal{M} \cup x).$$

We will often write $\mathrm{HOD}\,\mathcal{M} \cup X$ for $\mathrm{HOD}(\mathcal{M} \cup X)$.

Proposition 3.1 [1.10]. *Let X be a class for $\mathcal{N}$. If X is almost universal or X is definable from parameters in X, then $\mathrm{HOD}\,X$ is an inner model of $\mathcal{N}$.*

Proposition 3.2 [1.10]. *If $\mathcal{M}$ is an inner model of $\mathcal{N}$, then*
(i) $\mathrm{HOD}\,\mathcal{M} = \bigcup \{L[x]: x \subseteq \mathcal{M}\ \&\ x \in \mathrm{HOD}\,\mathcal{M}\}$,
(ii) $\mathcal{M} \vDash \mathrm{AC}$ *implies* $\mathrm{HOD}\,\mathcal{M} \vDash \mathrm{AC}$.

Definition 3.3 [2.13]. Let $\mathbb{B}$ be a c.b.a. in $\mathcal{M}$, A a c.b.s. of $\mathbb{B}$. Let G be $\mathbb{B}$-generic over $\mathcal{M}$, let H be A-generic over $\mathcal{M}$, $a \in \mathcal{M}[G]$ and $a \subseteq \mathcal{M}$. Then we define:
(i) $b \triangle c = (b - c) \vee (c - b)$,
(ii) $b^\wedge = \mathrm{Inf}\{x \in A: x \geqslant b\}$,
(iii) $b \sim_H c \equiv (b \triangle c)^\wedge \notin H$,
(iv) $\mathbb{B}/H = \{[b]_H: b \in \mathbb{B}\}$, where $[b]_H$ is the equivalence class of b under $\sim_H$.
(v) $\mathbb{B}(\bar{a}) =$ 'the smallest c.b.s. of $\mathbb{B}$ which includes $\{\|\check{x} \in \bar{a}\|: x \in \mathcal{M}\}$', $\bar{a}$ denotes a name of a.
(vi) $S(H) = \{b \in \mathbb{B}: b^\wedge \in H\}$.
(vii) $\mathbb{B}/a = \mathbb{B}/G \cap \mathbb{B}(\bar{a})$.

Proposition 3.3 [2.13 and 2.14]. *Under the hypotheses of Definition 3.3 the following hold:*
(i) *$G \cap A$ is A-generic over $\mathcal{M}$,*
(ii) *G is $S(G \cap A)$-generic over $\mathcal{M}[G \cap A]$,*
(iii) *If K is B/H-generic over $\mathcal{M}[H]$, and $G' = \{b \in \mathbb{B}: [b]_G \in K\}$, then G' is $\mathbb{B}$-generic over $\mathcal{M}$ and $\mathcal{M}[H][K] = \mathcal{M}[G']$.*
(iv) $\mathcal{M}[a] = \mathcal{M}[G \cap \mathbb{B}(\bar{a})]$,
(v) *If $\mathcal{M} \vDash \mathrm{AC}$ and $\mathcal{N} \vDash \mathrm{ZFC}$ and $\mathcal{M} \subseteq \mathcal{N} \subseteq \mathcal{M}[G]$, then there is a c.b.s. $\mathbb{D}$ of $\mathbb{B}$ s.t. $\mathcal{N} = \mathcal{M}[G \cap \mathbb{D}]$.*

Proposition 3.4 [3.8, 2.13 and 9.2]. (i) *If $\mathbb{B}$ is a c.b.a. in $\mathcal{M}$, then for any G $\mathbb{B}$-generic over $\mathcal{M}$, $\mathbb{B}/\mathbb{B}^* \cap G$ is a homogeneous c.b.a. in $\mathcal{M}[\mathbb{B}^* \cap G]$, $S(\mathbb{B}^* \cap G)$ is a homogeneous notion of forcing in $\mathcal{M}[\mathbb{B}^* \cap G]$.*

(ii) *If $\mathcal{N}$ is a model of ZF, $\mathcal{M}$ inner model of $\mathcal{N}$, $C \in \mathcal{N}$ a notion of forcing, $C \in (\mathrm{OD}\,\mathcal{M})^{\mathcal{N}}$, C is homogeneous in $\mathcal{N}$, then for any G C-generic over $\mathcal{N}$ $\mathrm{HOD}^{\mathcal{N}[G]} \subseteq (\mathrm{HOD}\,\mathcal{M})^{\mathcal{N}[G]} \subseteq (\mathrm{HOD}\,\mathcal{M})^{\mathcal{N}}$.*

Definition 3.4 [3.1]. (i) Aut $\mathbb{B}$ denotes the set of all automorphisms of a c.b.a. $\mathbb{B}$;

(ii) If $\sigma \in \operatorname{Aut}\mathbb{B}$ and $x \in \mathcal{M}$, let

$$\tilde{\sigma}(x) = (x - V\times\mathbb{B}) \cup \{(\tilde{\sigma}(y), \sigma b) : b \in \mathbb{B} \ \&\ (y, b) \in x\},$$

(iii) $\mathbb{B}^* = \{b \in \mathbb{B} : (\forall\sigma \in \operatorname{Aut}\mathbb{B})\ (\sigma b = b)\}$, $\mathbb{B}^*$ is called the *rigid part* of $\mathbb{B}$.

Theorem 3.5 (Vopěnka) [3.8]. *If G is $\mathbb{B}$-generic over $\mathcal{M}$, then* $(\mathrm{HOD}\,\mathcal{M})^{\mathcal{M}[G]} = \mathcal{M}[G \cap \mathbb{B}^*]$.

Proposition 3.6. *Let $\mathcal{M}$ be a c.t.m. of* ZF, *$\mathbb{B}_0$, $\mathbb{B}_1$-c.b.a.'s in $\mathcal{M}$, $\bar{\bar{\mathbb{B}}}_0 \leqslant \kappa$, $\mathbb{B}_0 \Vdash$ '$\check{\mathbb{B}}_1$ is (κ, ∞)-distributive', then*

$$(\mathrm{HOD}\,\mathcal{M})^{\mathcal{M}[H_0, H_1]} = \mathcal{M}[\mathbb{B}_0^* \cap H_0][\mathbb{B}_1^* \cap H_1],$$

for any $H_0 \times H_1$ which is $\mathbb{B}_0 \times \mathbb{B}_1$-generic over $\mathcal{M}$.

Proof. '$\supseteq$': The set $\{H_0' \cap \mathbb{B}_0^* : H_0'$ is $\mathbb{B}_0$-generic over $\mathcal{M}$ and (H_0'' is $\mathbb{B}_0$-generic over $\mathcal{M} \Rightarrow \mathcal{M}[H_0''] \subseteq \mathcal{M}[H_0'])\}$ is $\mathcal{M}$ definable in $\mathcal{M}[H_0, H_1]$.

H_0 is one of these H_0''s mentioned in the definition of the set, and for any such H_0' $\mathcal{M}[H_0] = \mathcal{M}[H_0']$. We use here the fact that H_0, H_0', H_0'' can be considered subsets of κ, therefore they all are from $\mathcal{M}[H_0]$ as H_1 does not add any new subsets of κ. By [3.5 Theorem 1] $\mathcal{M}[H_0] = \mathcal{M}[H_0']$ implies that there is $\sigma \in \operatorname{Aut}\mathbb{B}$ s.t. $H_0' = \sigma'' H_0$ but $\sigma \restriction \mathbb{B}_0^* = \mathrm{id}$, so the set has only one element, namely $\mathbb{B}_0^* \cap H_0$.

Now we know that $\mathbb{B}_0^* \cap H_0 \in \mathrm{HOD}\,\mathcal{M}$, and we are going to show that $\mathbb{B}_1^* \cap \mathrm{H}_1 \in \mathrm{HOD}\,\mathcal{M}$. It suffices to prove that $\mathcal{M}[H_1]$ is a definable subclass of $\mathcal{M}[H_0, H_1]$. $\mathcal{M}[H_0]$ is a definable subclass of $\mathcal{M}[H_0, H_1]$: $x \in \mathcal{M}[H_0]$ iff $(\exists H_0'\ \mathbb{B}_0$-generic over $\mathcal{M})$ $(\forall H_0''\ \mathbb{B}_0$-generic$)$ $(\mathcal{M}[\mathrm{H}_0''] \subseteq \mathcal{M}[\mathrm{H}_0'])$.

Claim. $x \in \mathcal{M}[H_1] \equiv (\exists H_1'\ \mathbb{B}_1$-generic over $\mathcal{M}[H_0]$ s.t. $V = \mathcal{M}[H_0][H_1']\ \&\ x \in \mathcal{M}[H_1])$ i.e. we claim that $\mathcal{M}[H_1]$ is definable in $\mathcal{M}[H_0, H_1]$. Note that it suffices to show the following.

Subclaim. If H_1' is $\mathbb{B}_1$-generic over $\mathcal{M}[H_0]$ and $\mathcal{M}[H_0][H_1'] = \mathcal{M}[H_0, H_1]$, then $H_1 \in \mathcal{M}[H_1']$. Because then by the symmetry argument $H_1' \in \mathcal{M}[H_1]$, and hence $\mathcal{M}[H_1] = \mathcal{M}[H_1']$, and the definition works.

Proof of the subclaim. We work in $\mathcal{M}[H_1'][H_0]$. Let $\boldsymbol{H}_1$ be a name for H_1 in $\mathcal{M}[H_1']$. Let $b_a = \|\check{a} \in \boldsymbol{H}_1\|_{\mathbb{B}_0}$ for $a \in H_1$. Since $\bar{\bar{\mathbb{B}}}_0 \leqslant \kappa$ we have at most κ different values of b_a. So let $\langle b_\eta : \eta < \kappa\rangle$ be these values, and let a_η be such that $\|\check{a}_\eta \in \boldsymbol{H}_1\|_{\mathbb{B}_0} = b_\eta$.

Because H_1' does not add any κ-sequences we know that $\langle a_\eta : \eta < \kappa\rangle \in \mathcal{M}[H_0]$. But then, because H_1 is $\mathbb{B}_1$-generic over $\mathcal{M}[H_0]$, $a = \prod_{\eta<\kappa} a_\eta \in H_1$. Let $\|\check{a} \in \boldsymbol{H}_1\|_{\mathbb{B}_0} = b \neq 0$, then $\|\check{a} \in \boldsymbol{H}_1\|_{\mathbb{B}_0} \leqslant \|\check{a}_\eta \in \boldsymbol{H}_1\|_{\mathbb{B}_0}$ i.e. $0 \neq b \leqslant b_\eta$ for all $\eta < \kappa$. Now, for any c, if $c \in H_1$, then $\|\check{c} \in \boldsymbol{H}_1\| = b_\eta$ for some $\eta < \kappa$, and then $b \Vdash \check{c} \in \boldsymbol{H}_1$, as $b \leqslant b_\eta$. Therefore $H_1 = \{c : b \Vdash \check{c} \in \boldsymbol{H}_1\}$, and $H_1 \in \mathcal{M}[H_1']$.

'$\subseteq$': By Theorem 3.5, it suffices to show that the way from $\mathcal{M}[\mathbb{B}_0^* \cap H_0][\mathbb{B}_1^* \cap H_1]$ to $\mathcal{M}[H_0, H_1]$ is generic via a homogeneous notion of forcing.

H_0 is $S(H_0 \cap \mathbb{B}_0^*)$-generic over $\mathcal{M}[\mathbb{B}_0^* \cap H_0][\mathbb{B}_1^* \cap H_1]$, because H_0 is $S(H_0 \cap \mathbb{B}_0^*)$-generic over $\mathcal{M}[\mathbb{B}_0^* \cap H_0][H_1]$.

Also $\mathcal{M}[\mathbb{B}_0^* \cap H_0][\mathbb{B}_1^* \cap H_1][H_0] = \mathcal{M}[H_0][\mathbb{B}_1^* \cap H_1]$.

H_1 is $S(\mathbb{B}_1^* \cap H_1)$-generic over $\mathcal{M}[H_0][\mathbb{B}_1^* \cap H_1]$. ($H_1$ is $S(\mathbb{B}_1^* \cap H_1)$-generic over $\mathcal{M}[\mathbb{B}_1^* \cap H_1]$, also H_0 is $\mathbb{B}_0$-generic over $\mathcal{M}[H_1]$. So, by the product lemma, H_1 is $S(\mathbb{B}_1^* \cap H_1)$-generic over $\mathcal{M}[H_0][\mathbb{B}_1^* \cap H_1]$.) So, by the product lemma, $H_0 \times H_1$ is $S(H_0 \cap \mathbb{B}_0^*) \times S(H_1 \cap \mathbb{B}_1^*)$-generic over $\mathcal{M}[H_0 \cap \mathbb{B}_0^*][H_1 \cap \mathbb{B}_1^*]$. By Proposition 3.4(i) each $S(H_i \cap \mathbb{B}_i^*)$, $i = 0, 1$, is homogeneous, so is their product.

Proposition 3.7. *If C is a class of conditions in $\mathcal{M}$, then for any G C-generic over $\mathcal{M}$*

$$(\mathrm{HOD}\,\mathcal{M})^{\mathcal{M}[G]} = \bigcup_{\gamma < On} \mathcal{M}[G \cap \mathrm{r.o.}(C_\gamma)^*].$$

Proof. '$\supseteq$': Exactly as in the proof of '$\supseteq$' in Proposition 3.6.

'$\subseteq$': If $x \in \mathrm{HOD}\,\mathcal{M}$, then there is β s.t. x is hereditarily ordinal definable over $(V_\beta, \in)$, possibly with parameters from $\mathcal{M}$. Since there is γ s.t.

$$V_\beta^{\mathcal{M}[G]} = V_\beta^{\mathcal{M}[G \cap \mathrm{r.o.}(C_\gamma)]}$$

(C satisfies Easton's conditions, Section 1.2), by Theorem 3.5,

$$x \in \mathcal{M}[G \cap \mathrm{r.o.}(C_\gamma)^*].$$

3.1. The derivative of $(B_\alpha)_\lambda$

Definition 3.5 [3.9]. Let E be a formula of $\mathcal{L}_{\mathrm{ZF}}(U)$, and $\mathcal{M}$ be the interpretation of U.

(i) $\varphi_E(x, \alpha, F, u)$ stands for 'F is a formula of $\mathcal{L}_{\mathrm{ZF}}$, and $V_\alpha \cap \{y: E(x, y)\} \models F(u)$',

(ii) If B is a c.b.a. in $\mathcal{M}$, $t \in \mathcal{M}$, then we put

$B^{E,t}$ = 'the c.b.s. of B generated by $\{\|\varphi_E(t, \check{\alpha}, \check{F}, \check{u})\|: \alpha \in \mathrm{On}$, F—formula of $\mathcal{L}_{\mathrm{ZF}}$, $u \in \mathcal{M}\}$'.

Assume until the end of Section 3.1, that B is a c.b.a. in $\mathcal{M}$ and G is B-generic over $\mathcal{M}$.

Proposition 3.8 [3.9]. *If $X = \{y: \mathcal{M}[G] \models E(x, y)\}$ is an inner model of $\mathcal{M}[G]$, $\mathcal{M} \subseteq X$, $t \in \mathcal{M}$ and* $\mathrm{val}_G(t) = x$, *then* $(\mathrm{HOD}\,\mathcal{M})^X = \mathcal{M}[G \cap B^{E,t}]$.

Note. If E is '$x = x$', then $B^{E,\emptyset} = B^*$.

Let $(B_\alpha)_{\alpha<\lambda}$ be a decreasing family of c.b.s. of B. Let $X = \bigcap \{\mathcal{M}[G < B_\alpha]: \alpha < \lambda\}$. Then X is an inner model of $\mathcal{M}[G]$ (by [3.9]). X is defined by the following formula:

$$E(v, w) \equiv (\exists f)(\exists x)(\exists y)(v = (y, x, f) \;\&\; \mathrm{func}(f) \;\&\; \mathrm{dom}(f) = x \;\&\; (\forall \alpha \in x)(w \in U[y \cap f(\alpha)])).$$

Clearly, U plays role of $\mathcal{M}$, $f(\alpha) = B_\alpha$, $x = \lambda$.

Definition 3.6. Let everything be as above.

$$\Gamma_{B_0} = \{(\check{a}, a) : a \in B_0\},$$

t_0 a name s.t. $\|t_0 = (\Gamma_{B_0}, \check{\lambda}, \widetilde{(B_\alpha)_{\alpha<\lambda}})\| = \mathbb{1}$.

We define the *derivative* of the family $(B_\alpha)_{\alpha<\lambda}$ to be B_0^{E,t_0}. We denote it by $(B_\alpha)^*_\lambda$.

Proposition 3.9 [3.9 and 3.10]. (i) $(\text{HOD}\,\mathcal{M})^X = \mathcal{M}[G \cap (B_\alpha)^*_\lambda]$,
(ii) $(B_\alpha)^*_\lambda \subseteq \bigcap_{\alpha<\lambda} B_\alpha$.

Definition 3.7. $B^0 = B$, $B^{\alpha+2} = (B^{\alpha+1})^*$, $B^{\lambda+1} = (B^{\alpha+1})^*_\lambda$.

Theorem 3.10 [3.10 and 10.2]. *Let* $X_0 = \mathcal{M}[G]$,

$$X_{\alpha+1} = \mathcal{M}[G \cap B^{\alpha+1}], \qquad X_\lambda = \bigcap \{\mathcal{M}[G \cap B^{\alpha+1}] : \alpha < \lambda\}.$$

Then X_α's form a decreasing definable family of inner models of $\mathcal{M}[G]$, $X_{\alpha+1} = (\text{HOD}\,\mathcal{M})^{X_\alpha}$ for all α, and $X_\lambda = \bigcap\{X_\alpha : \alpha < \lambda\}$ for all limit λ.

Note. We shall often write $(\text{HOD}^\alpha\mathcal{M})^{\mathcal{M}[G]}$ instead of X_α.

Corollary 3.11. *If $\lambda < \kappa^+$ and B is (κ, ∞)-distributive we can define $B^\lambda = \bigcap_{\alpha<\lambda} B^\alpha$, and then*

$$\text{HOD}^\lambda\mathcal{M} = \mathcal{M}[G \cap B^\lambda], \quad \textit{also} \quad (B^\alpha)^*_\lambda = (B^\lambda)^*.$$

Proof. The result follows easily by Proposition 1.7 and Corollary 1.8.

Corollary 3.12. *If $\bar{\bar{B}} \leqslant \kappa$, then the sequence $\text{HOD}^\alpha\mathcal{M}$ is constant for $\alpha \geqslant \kappa^+$.*

Proof. $\text{HOD}^\beta\mathcal{M} = \text{HOD}^{\beta+1}\mathcal{M}$ implies that the sequence is constant for $\alpha \geqslant \beta$. Also $\text{HOD}^{\alpha+1}\mathcal{M} = \mathcal{M}[G \cap B^{\alpha+1}]$, and $B^{\alpha+1} \subseteq B$. Since $\alpha < \beta$ implies $B^{\alpha+1} \supseteq B^{\beta+1}$, the sequence cannot descend more than κ^+ steps.

Proposition 3.13. *Assume that B_0, B_1 are c.b.a.'s s.t. $\bar{\bar{B}}_0 \leqslant \kappa$, B_1 is κ'-closed for some $\kappa' \geqslant \kappa$, $\lambda < (\kappa')^+$. Assume also that $H_0 \times H_1$ is $B_0 \times B_1$-generic over $\mathcal{M}$, and for all $\alpha < \lambda$*

$$(\text{HOD}^{\alpha+1}\mathcal{M})^{\mathcal{M}[H_0 \times H_1]} = \mathcal{M}[B_0^{\alpha+1} \cap H_0][B_1^{\alpha+1} \cap H_1].$$

Then

$$(\text{HOD}^\lambda\mathcal{M})^{\mathcal{M}[H_0 \times H_1]} = \bigcap_{\alpha<\lambda} \mathcal{M}[B_0^{\alpha+1} \cap H_0][B_1^\lambda \cap H_1].$$

Proof. '$\supseteq$': For any $\lambda' \leqslant \lambda$ $B_1^{\lambda'} = \bigcap_{\alpha<\lambda'} B_1^{\alpha+1}$ is a c.b.s. of any $B_1^{\alpha+1}$. Thus

$$\mathcal{M}[B_0^{\alpha+1} \cap H_0][B_1^\lambda \cap H_1] \subseteq \mathcal{M}[B_0^{\alpha+1} \cap H_0][B_1^{\alpha+1} \cap H_1].$$

'$\subseteq$': If $\alpha_0 < \lambda$, then

$$\mathrm{HOD}^{\alpha_0+1}\mathcal{M} \subseteq \mathcal{M}[B_0^{\alpha_0+1} \cap H_0][B_1^{\alpha_0+1} \cap H_1].$$

For $\lambda > \beta > \alpha_0$

$$\mathrm{HOD}^{\beta+1}\mathcal{M} \subseteq \mathcal{M}[B_0^{\alpha_0+1} \cap H_0][B_1^{\beta+1} \cap H_1],$$

so

$$\mathrm{HOD}^{\lambda}\mathcal{M} \subseteq \bigcap_{\beta<\lambda} \mathcal{M}[B_0^{\alpha_0+1} \cap H_0][B_1^{\beta+1} \cap H_1].$$

By Proposition 1.7 and Corollary 1.8 the right-hand side equals $\mathcal{M}[B_0^{\alpha_0+1} \cap H_0] \times [B_0^{\lambda} \cap H_1]$. Finally

$$\mathrm{HOD}^{\lambda}\mathcal{M} \subseteq \bigcap_{\alpha_0<\lambda} \mathcal{M}[B_0^{\alpha_0+1} \cap H_0][B_1^{\lambda} \cap H_1].$$

3.2

The following part contains a version of Theorem 1 [4.9], which we will use in the next chapter.

Definition 3.8. Let a be a set in $\mathcal{M}$.

$$C(a) = \{f: \mathrm{func}(f)\ \&\ \mathrm{dom}\, f \subseteq \omega\ \&\ \bar{\bar{f}} < \omega\ \&\ \mathrm{rng}\, f \subseteq a\}.$$

$C(a)$ is called the *collapsing algebra* generated by a.

Note. $C(a)$ ordered by inverse inclusion is a notion of forcing which collapses a onto ω.

Theorem 3.14. *Let $\mathcal{M} \models \mathrm{ZFC}$, $\rho < \mathrm{On}$, $y \in \mathcal{M}$ and $L_\rho[y] \models \cdot \mathrm{ZF}^-$. Assume also $y \in L_\rho[y]$ and $\rho \geq \bar{\bar{V}}_{\mathrm{rank}\, y}$. Let $\langle C, \leq\rangle \in L_\rho[y]$ and let G be C-generic over $\mathcal{M}$. If H is $C(\rho)$-generic over $\mathcal{M}(G)$, then there exists K which is $C(\rho)$-generic over $\mathcal{M}$ and $\mathcal{M}[G][H] = \mathcal{M}[K]$.*

Proof. There is a canonical isomorphism $C(\rho) \times C(\rho) \simeq C(\rho)$, so there are H_0, H_1 s.t. $H_0 \times H_1$ is $C(\rho) \times C(\rho)$-generic over $\mathcal{M}[G]$ and $\mathcal{M}[G][H] = \mathcal{M}[G][H_0][H_1]$. Also $\mathcal{M}[G, H_0] \models \bar{\bar{\rho}} = \omega$, and $\mathcal{M}[G, H_0] \models \overline{\overline{L_\rho[y][G]}} = \omega$. Since $L_\rho[y] \models \mathrm{ZF}^-$, $L_\rho[y][G] = \mathrm{val}_G(L_\rho[y])$.

By the product lemma (Proposition 1.2), H_1 is $C(\rho)$-generic over $\mathcal{M}[G, H_0]$, and (since $\bar{\bar{\rho}} = \omega$) there is $i: C(\rho) \xrightarrow{\sim} C(\omega)$ s.t. $H' = i(H_1)$ is $C(\omega)$-generic over $\mathcal{M}[G, H_0]$ and $\mathcal{M}[G\ H_0][H'] = \mathcal{M}[G, H_0, H_1]$.

Applying again the product lemma:

$$\mathcal{M}[G, H_0][H'] = \mathcal{M}[H_0][G][H'] = \mathcal{M}[H_0][H'][G] = \mathcal{M}[H_0, H'][G].$$

C is countable in $\mathcal{M}[H_0, H']$. By Remark 4.5.2 [3], any separative and atomless countable C contains a dense subset isomorphic to $C(\omega)$, hence r.o.$(C) =$

r.o.$(C(\omega))$. Therefore there is G' $C(\omega)$-generic over $\mathcal{M}[H_0, H']$ s.t. $\mathcal{M}[H_0, H'] \times [G] = \mathcal{M}[H_0, H'][G']$.

Finally $\mathcal{M}[G, H] = \mathcal{M}[H_0, H', G']$ and K, defined as $H_0 \times H' \times G'$, is $C(\rho) \times C(\omega) \times C(\omega)$-generic over $\mathcal{M}$ (by the product lemma). Since $C(\rho) \simeq C(\rho) \times C(\omega) \times C(\omega)$, the theorem is proven.

3.3

Theorem 1 [5.1] has to be quoted here, because it will be used later.

Assume that B is a c.b.a. in $\mathcal{M}$, G is B-generic over $\mathcal{M}$, $x \in \mathcal{M}[G]$, $\bar{x}$ is a name for x (i.e. $\mathrm{val}_G(\bar{x}) = x$), $\bar{\bar{B}} \leq \kappa$. Let $\mathcal{G} = \{\sigma''G : \sigma \in \mathrm{Aut}\, B\}$.

Definition 3.9. For $H \in \mathcal{G}$ we define:

$$T'(H) = \{(t, \mathrm{val}_H(t)) : t \in \mathcal{M} \;\&\; \mathrm{rank}(t) \leq \mathrm{rank}(\bar{x}) \;\&\; \|t \in \mathrm{TC}\{\bar{x}\}\| \in H\},$$

$$T(H) = \{(b, Z) : b \in H \;\&\; Z \subseteq T'(H) \;\&\; \bar{\bar{Z}} < \omega\},$$

$$T = \bigcup \{T(H) : H \in \mathcal{G} \;\&\; \mathrm{val}_H(\bar{x}) = x\}.$$

Theorem 3.15. (i) $\mathcal{M}[T] = (\mathrm{HOD}\, \mathcal{M}[x])^{\mathcal{M}[G]}$,

(ii) *$\langle T, \leq \rangle$ is a homogeneous notion of forcing in $\mathcal{M}[T]$, $T(G)$ is T-generic over $\mathcal{M}[T]$ and $\mathcal{M}[G] = \mathcal{M}[T][T(G)]$, where $(b, Z) \leq_T (b', Z') \equiv b \leq b' \;\&\; Z \supseteq Z'$.*

Remark 3.16. $T \in V_\beta \cap \mathcal{M}[G]$, where $\beta = \max(\mathrm{rank}(\bar{x}), \mathrm{rank}(B)) + 11$.

3.4

The next part is devoted to the proof of the main theorem of Chapter 3. This theorem states that there are not too many classes $\mathrm{OD}(\mathcal{M} \cup x)$ in a generic extension of $\mathcal{M}$ via a c.b.a. $B \in \mathcal{M}$. We shall prove that there are at most $2^{2^{\bar{\bar{B}}}}$ such classes.

Definition 3.10 [7.2 and 7.3]. For $x \in \mathcal{M}$ we set

$$\Omega(x) = \{\sigma \in \mathrm{Aut}\, B : \|\sigma(x) = x\| = \mathbb{1}\}.$$

Definition 3.11. (i) For $b \in B$, let $T(b) = \{\sigma \in \mathrm{Aut}\, B : (\forall a \leq \mathbb{1} - b)(\sigma a = a)\}$.

(ii) For $x, y \in \mathcal{M}$, let $(x, y)^B$ denote the canonical term z s.t. $\|z = (x, y)\| = \mathbb{1}$.

Definition 3.12. If $\Omega \subseteq \mathrm{Aut}\, B$ and $y \in \mathcal{M}$, we define

$$\|\Omega, y\| = \sup\{\|z_1 \in y\| \wedge \cdots \wedge \|z_n \in y\| : z_i \in \mathrm{TC}(y) \;\&\; \Omega((z_1, \ldots, z_n)^B) \subseteq \Omega\}.$$

Let $\bar{\Omega}$ denote the closure of $\Omega \subseteq \mathrm{Aut}\, B$. (For the definition see [7.1].) $\Omega \subseteq \mathrm{Aut}\, B$ is said to be closed (subgroup) iff $\Omega = \bar{\Omega}$.

Proposition 3.17 [7.2]. (i) $\Omega((x, y)^B) = \Omega(x) \cap \Omega(y)$,

(ii) *If we define* $\Gamma(\Omega) = \{(\tilde{\sigma}(\Gamma_B), \mathbb{1}): \sigma \in \Omega\}$, *then* $\Omega(y) = \Omega(\Gamma(\Omega(y)))$,

(iii) $\Omega(\check{z}) = \operatorname{Aut} B$ *for all* $z \in \mathcal{M}$.

Note. Γ_B was introduced in Definition 3.6.

Theorem 3.18 [7.3]. *For all* $x, y \in \mathcal{M}$,

$$\|x \in \mathrm{OD}\ \mathcal{M}\{y\}\| = \sup\{b: \Omega(x) \supseteq \Omega(y) \cap T(b)\}.$$

Theorem 3.19 [7.3]. *For all* $x, y \in \mathcal{M}$,

$$\|x \in \mathrm{OD}\ \mathcal{M}y\| = \sup\{b \wedge \|\Omega, y\|: \Omega \textit{ is a closed subgroup of } \operatorname{Aut} B \textit{ and } \Omega(x) \supseteq \Omega \cap T(b)\}.$$

Note. 'Closed subgroup' will be abbreviated as c.s.g.

Fact 3.20 [7.3]. $\|x \in \mathrm{OD}\ \mathcal{M}y \equiv (\exists z_1, \ldots, z_n \in y)(x \in \mathrm{OD}\ \mathcal{M}\{(z_1 \cdots z_n)^B\}\| = \mathbb{1}$.

Proposition 3.21 [7.4]. *Let* y, z *be terms in* $\mathcal{M}$, *assume also that* $\|\Omega, y\| = \|\Omega, z\|$ *for all c.s.g.* $\Omega \subseteq \operatorname{Aut} B$. *Then*

$$(\mathrm{OD}\ \mathcal{M} \cup \mathrm{val}_G(y))^{\mathcal{M}[G]} = (\mathrm{OD}\ \mathcal{M} \cup \mathrm{val}_G(z))^{\mathcal{M}[G]},$$

for any B*-generic over* $\mathcal{M}$ *set* G.

The next theorem strengthens a bit the results of Grigorieff.

Theorem 3.22 (cf. [7.4]). *Let* B *be a c.b.a. in* $\mathcal{M}$, *where* $\mathcal{M}$ *satisfies* ZFC. *Then for any* $x_0 \in \mathcal{M}[G]$, *where* G *is* B*-generic over* $\mathcal{M}$, *there exists* $y_0 \in \mathcal{M}[G]$ *s.t.* $\overline{\overline{\mathrm{TC}(y_0)}} \leq 2^{\bar{\bar{B}}}$, *and* $(\mathrm{OD}\ \mathcal{M} \cup y_0)^{\mathcal{M}[G]} = (\mathrm{OD}\ \mathcal{M} \cup x_0)^{\mathcal{M}[G]}$.

Proof. Proposition 3.21 is the key to the proof. If x is a name for x_0, then we will construct a name y for y_0 s.t. $\|\Omega, y\| = \|\Omega, x\|$ for all c.s.g. Ω of $\operatorname{Aut} B$. Since $\overline{\overline{\mathrm{TC}(y)}} \leq 2^{\bar{\bar{B}}}$ implies $\overline{\overline{\mathrm{TC}(\mathrm{val}_G(y))}} \leq 2^{\bar{\bar{B}}}$, it will suffice to find y with the above properties.

The construction of y will be done in two steps.

Let $y' = \{((\Gamma(\Omega(z)), \check{z})^B, \|z \in x\|): z \in \operatorname{dom} x\}$, and let $z' = (\Gamma(\Omega(z)), \check{z})^B$ for $z \in \operatorname{dom} x$.

Claim. $\|\Omega, y'\| = \|\Omega, x\|$ for all c.s.g. of $\operatorname{Aut} B$.

Proof of the claim. It suffices to show that $\|z \in x\| = \|z' \in y'\|$ and notice that Proposition 3.17 implies that

$$\Omega((\Gamma(\Omega(z)), \check{z})^B) = \Omega(\Gamma(\Omega(z))) \cap \Omega(\check{z}) = \Omega(z) \cap \operatorname{Aut} B = \Omega(z).$$

We evaluate $\|z' \in y'\| = \sum_{w \in \mathrm{dom}\, y'} y'(w)\, \|w = z'\|$. dom y' consists of pairs $(\Gamma(\Omega(t)), \check{t})^B$, $t \in \mathrm{dom}\, x$. Therefore

$$\|w = z'\| = \|\Gamma(\Omega(t)) = \Gamma(\Omega(z))\ \&\ \check{t} = \check{z}\|,$$

where $w = (\Gamma(\Omega(t)), \check{t})^B$. But the value

$$\|\check{t} = \check{z}\| = \begin{cases} \mathbb{0}, & \text{if } \check{t} \neq \check{z}, \\ \mathbb{1}, & \text{if } \check{t} = \check{z}, \end{cases}$$

hence $\|z' \in y'\| = y'(z') = \|z \in x\|$. And this proves the claim.

We can enumerate the elements of y', so $y' = \{(\Gamma(\Omega(z_i)), \check{z}_i)^B, b_i) : i < \delta\}$, where δ is a cardinal number and b_i's belong to B.

For fixed $\Gamma(\Omega(z_{i_0}))$ we can uniformize the relation $\mathbf{A}_{i_0}^{-1}$, where $\mathbf{A}_{i_0} = \{(\Gamma(\Omega(z_{i_0})), \check{z}_i)^B, b_i) \in y'\}$. So we obtain $\underset{\sim}{\mathbf{A}}_{i_0} \subseteq \mathbf{A}_{i_0}$ s.t. for every b_i there is exactly one $(\Gamma(\Omega(z_{i_0})), \check{z}_i)^B$ s.t. the pair $((\Gamma(\Omega z_{i_0})), \check{z}_i)^B, b_i) \in y'$.

We set

$$A_{i_0} = \{((\Gamma(\Omega(z_{i_0})), \check{b}_i)^B, b_i) : ((\Gamma(\Omega(z_{i_0})), \check{z}_i)^B, b_i) \in \underset{\sim}{\mathbf{A}}_{i_0}\}$$

We finally define $y = \bigcup_{i_0 < \delta} A_{i_0}$. Then $y \subseteq \{(v, t)^B : v \subseteq \mathrm{Aut}\, B \times \{\mathbb{1}\}, t \in B\} \times B$, therefore the power of TC(y) is less than $2^{\bar{\bar{B}}}$.

Now, it suffices to show that $\|\Omega, y\| = \|\Omega, y'\|$ for all c.s.g. Ω of Aut B. This is done by the same argument as in the case of y'. Let $t_i = (\Gamma(\Omega(z_i)), \check{b}_i)^B$, then $\|t_i \in y\| = b_i = \|z_i' \in y'\|$, because we have uniformized $\mathbf{A}_i^{-1}$. As previously

$$\Omega(t_i) = \Omega((\Gamma(\Omega(z_i)), \check{b}_i)^B) = \Omega(z_i) = \Omega(z_i').$$

Corollary 3.23. *If $\bar{\bar{B}} \leq \kappa$, then for any $t \in \mathcal{M}[G]$, there is $u \in V_{\kappa+10} \cap \mathcal{M}[G]$, $u \subseteq V_{\kappa+10} \cap (\mathrm{HOD}\, \mathcal{M} \cup t)^{\mathcal{M}[G]}$ s.t.*

$$(\mathrm{HOD}\, \mathcal{M} \cup t)^{\mathcal{M}[G]} = (\mathrm{HOD}\, \mathcal{M} \cup u)^{\mathcal{M}[G]}.$$

Proof. Put $x_0 = t$, then $u = y_0$ (given by the theorem) belongs to $V_{\kappa+10} \cap \mathcal{M}[G]$. Clearly, OD $\mathcal{M} \cup u$ = OD $\mathcal{M} \cup t$ implies HOD $\mathcal{M} \cup u$ = HOD $\mathcal{M} \cup t$. u is contained in $V_{\kappa+10} \cap (\mathrm{HOD}\, \mathcal{M} \cup t)^{\mathcal{M}[G]}$ by the Definition 3.2 of HOD $\mathcal{M} \cup x$.

Corollary 3.24. *If $X \subseteq \mathcal{M}[G]$ is a class for $\mathcal{M}[G]$, $\bar{\bar{B}} \leq \kappa$ (in particular, if $X \in \mathcal{M}[G]$), then*

$$\mathrm{HOD}(\mathcal{M} \cup X) = \mathrm{HOD}(\mathcal{M} \cup (V_{\kappa+10} \cap \mathrm{HOD}(\mathcal{M} \cup X))).$$

Proof (cf. [7.4, Lemma 1]). By the replacement we have

$$\mathrm{HOD}\, \mathcal{M} \cup X = \bigcup \{\mathrm{HOD}\, \mathcal{M} \cup (X \cap V_\alpha) : \alpha < \mathrm{On}\}.$$

Applying Corollary 3.23: for $t = X \cap V_\alpha$ we can find $u \subseteq V_{\kappa+10}$ s.t. HOD $\mathcal{M} \cup (X \cap V_\alpha)$ = HOD $\mathcal{M} \cup u$, also $u \subseteq V_{\kappa+10} \cap \mathrm{HOD}\, \mathcal{M} \cup (X \cap V_\alpha)$. Thus

$$\mathrm{HOD}\, \mathcal{M} \cup X = \bigcup \{\mathrm{HOD}\, \mathcal{M} \cup ((\mathrm{HOD}\, \mathcal{M} \cup (X \cap V_\alpha)) \cap V_{\kappa+10}) : \alpha < \mathrm{On}\}.$$

Clearly,

$$(\mathrm{HOD}\,\mathcal{M}\cup(X\cap V_\alpha))\cap V_{\kappa+10}\subseteq(\mathrm{HOD}\,\mathcal{M}\cup X)\cap V_{\kappa+10},$$

and the corollary follows.

Corollary 3.25 (cf. [7.4, Theorem 2). *Under the above assumptions: For every* $t\in\mathcal{M}[G]$ *there is* $u\in V_{\kappa+11}\cap\mathcal{M}[G]$ *s.t.* $\mathrm{HOD}\,\mathcal{M}[t]=\mathrm{HOD}\,\mathcal{M}[u]$.

Proof. $\mathrm{HOD}\,\mathcal{M}[t]=\mathrm{HOD}\,\mathcal{M}\cup\mathrm{TC}(\{t\})$ is an inner model of $\mathcal{M}[G]$ (by Proposition 3.1), $V_{\kappa+10}\cap\mathrm{HOD}\,\mathcal{M}[t]\in\mathrm{HOD}\,\mathcal{M}[t]$, and $\mathrm{HOD}\,\mathcal{M}[t]=\mathrm{HOD}\,\mathcal{M}[V_{\kappa+10}\cap\mathrm{HOD}\,\mathcal{M}[t]]$ by Corollary 3.24.

3.5

The theorem which appears below corresponds to Grigorieff's Theorem 4 [9.3].

Theorem 3.26. *Let G be B-generic over $\mathcal{M}$, $\bar{\bar{B}}\leqslant\kappa$, $\mathcal{M}\models\mathrm{ZFC}$. Let $X\subseteq\mathcal{M}[G]$, $X\models\mathrm{ZF}$, $X=(\mathrm{HOD}\,\mathcal{M}\cup X)^{\mathcal{M}[G]}$.*

Then X is a class for $\mathcal{M}[G]$ and there is $x\in V_{\kappa+22}\cap X$ s.t. $X=\mathcal{M}[x]$.

Proof. X is a class for $\mathcal{M}[G]$ by Theorem 3 [9.1], hence $X\cap V_\alpha\cap\mathcal{M}[G]\in\mathcal{M}[G]$. Since $X=\bigcup\{X\cap V_\alpha:\alpha<\mathrm{On}\}$, and

$$X\subseteq\bigcup\{\mathrm{HOD}\,\mathcal{M}[X\cap V_\alpha]:\alpha<\mathrm{On}\}\subseteq\mathrm{HOD}\,\mathcal{M}\cup X=X.$$

By the proof of Corollary 3.25,

$$\mathrm{HOD}\,\mathcal{M}[X\cap V_\alpha]=\mathrm{HOD}\,\mathcal{M}[V_{\kappa+10}\cap\mathrm{HOD}\,\mathcal{M}[V_\alpha\cap X]].$$

So we have that

$$X=\bigcup\{\mathrm{HOD}\,\mathcal{M}[X\cap V_\alpha]:\alpha<\mathrm{On}\}\subseteq\mathrm{HOD}\,\mathcal{M}[X\cap V_{\kappa+11}]\subseteq X.$$

Therefore $X=(\mathrm{HOD}\,\mathcal{M}[X\cap V_{\kappa+11}])^{\mathcal{M}[G]}$.

By Theorem 3.15 and Remark 3.16;

$$(\exists T\in V_\beta\cap X)(\mathrm{HOD}\,\mathcal{M}[X\cap V_{\kappa+11}])^{\mathcal{M}[G]}=\mathcal{M}[T],$$

where

$$\beta=\max(\mathrm{rank}(X\cap V_{\kappa+11}),\mathrm{rank}(B))+11=\kappa+22.$$

3.6

Remark 3.27. Let B_0, B_1 be c.b.a.'s in $\mathcal{M}\models\mathrm{AC}$, B a c.b.s. of B_1. Assume that $\bar{\bar{B}}_0\leqslant\kappa$, and B_1 is κ-closed. Then B is (κ,∞)-distributive in any submodel $\mathcal{M}[x]$ of $\mathcal{M}[H_0]$ $(x\in\mathcal{M}[H_0])$, where H_0 is B_0-generic over $\mathcal{M}$.

Proof. B_1 can be considered as a subset of On. The (κ,λ)-distributivity of B is defined as:

$$\prod_{\alpha<\kappa}\sum_{\beta<\lambda} u_{\alpha\beta}=\sum_{f\in{}^{\kappa}\lambda}\prod_{\alpha<\kappa} u_{\alpha,f(\alpha)},$$

where $u_{\alpha\beta}$'s belong to B. Let $U=\{u_{\alpha\beta}: \beta<\lambda, \alpha<\kappa\}$ be an element of $\mathcal{M}[x]$, $\mathcal{M}[U]\models \mathrm{AC}$, hence by Proposition 3.3 $\mathcal{M}[U]=\mathcal{M}[A\cap H_0]$ for some c.b.s A of B_0, but then, by the Easton's lemma (Proposition 1.10), B_1 is still (κ,∞)-distributive over $\mathcal{M}[U]$, so for all $\lambda\in \mathrm{On}$

$$\mathcal{M}[U]\models\prod_{\alpha<\kappa}\sum_{\beta<\lambda} u_{\alpha\beta}=\sum_{f\in{}^{\kappa}\lambda}\prod_{\alpha<\kappa} u_{\alpha,f(\alpha)},$$

because B is a c.b.s. of B_1. Thus

$$\mathcal{M}[x]\models\prod_{\alpha<\kappa}\sum_{\beta<\lambda} u_{\alpha\beta}\leqslant\sum_{f\in{}^{\kappa}\lambda}\prod_{\alpha<\kappa} u_{\alpha,f(\alpha)}.$$

Since the opposite inequality always holds, the remark is proven.

Theorem 3.28. *Let* B_0, B_1 *be c.b.s.'s in* $\mathcal{M}\models \mathrm{ZF}+V=L$, $\bar{\bar{B}}_0\leqslant\kappa$, B_1 *is* $\kappa^*=\omega_{\omega_{\kappa+22}+12}$*-closed. Assume that* $H_0\times H_1$ *is* $B_0\times B_1$*-generic over* $\mathcal{M}$ *and, for* $\alpha<\lambda<\kappa^*$,

$$(\mathrm{HOD}^{\alpha+1})^{\mathcal{M}[H_0]}=\mathcal{M}[H_0\cap B_0^{\alpha+1}]$$

and

$$(\mathrm{HOD}^{\alpha+1}\mathcal{M}[H_1\cap B_1^{\lambda+1}])^{\mathcal{M}[H_0,H_1\cap B_1^{\lambda+1}]}=\mathcal{M}[H_1\cap B_1^{\lambda+1}][H_0\cap B_0^{\alpha+1}].$$

Assume also that there is y_λ *s.t.* $\mathrm{rank}(y_\lambda)\leqslant\kappa+22$ *and*

$$(\mathrm{HOD}^{\lambda})^{\mathcal{M}[H_0]}=\mathcal{M}[y_\lambda],$$

$$(\mathrm{HOD}^{\lambda})^{\mathcal{M}[H_0,H_1]}=\mathcal{M}[y_\lambda][H_1\cap B_1^{\lambda}]$$

and

$$(\mathrm{HOD}^{\lambda}\mathcal{M}[H_1\cap B_1^{\lambda}])^{\mathcal{M}[H_0,H_1\cap B_1^{\lambda}]}=\mathcal{M}[y_\lambda][H_1\cap B_1^{\lambda}].$$

Then there is $y_{\lambda+1}$ *s.t.*

$$(\mathrm{HOD}^{\lambda+1})^{\mathcal{M}[H_0]}=\mathcal{M}[y_{\lambda+1}]$$

and

$$(\mathrm{HOD}^{\lambda+1})^{\mathcal{M}[H_0,H_1]}=\mathcal{M}[y_{\lambda+1}][B_1^{\lambda+1}\cap H_1].$$

Proof. Let $y_{\lambda+1}$ be s.t. $(\mathrm{HOD}^{\lambda+1})^{\mathcal{M}[H_0]}=\mathcal{M}[y_{\lambda+1}]$. We show below that the second equality of the conclusion of the theorem holds.

'$\supseteq$': $\mathcal{M}[y_\lambda]=\mathcal{M}[V_{\kappa+23}]$ is a definable subclass in $\mathcal{M}[y_\lambda][H_1\cap B_1^{\lambda}]$. As in Proposition 3.6 we can argue that $H_1\cap B_1^{\lambda+1}$ is definable. We only need to notice that $\mathcal{M}[H_1\cap B_1^{\lambda}]$ is a definable subclass of $\mathcal{M}[y_\lambda][H_1\cap B_1^{\lambda}]$. We give some details below:

By Theorem [6.1] there is a notion of forcing $\mathbb{D}$ s.t. $\mathcal{M}[H_0]$ is a generic extension of $\mathcal{M}[y_\lambda]$ via $\mathbb{D}$.

Claim. Rank of $\mathbb{D}$ is less or equal $\omega_{\kappa+22}+12$.

We will prove the claim later. Now, assume it is true. Then $x\in\mathcal{M}[H_1\cap B_1^\lambda]$ iff $(\exists H'\ B_1^\lambda$-generic over $\mathcal{M}[y_\lambda])\times(x\in\mathcal{M}[H']\ \&\ \mathcal{M}[y_\lambda][H']=\mathcal{M}[y_\lambda][H_1\cap B_1^\lambda])$. This gives the definition of $\mathcal{M}[H_1\cap B_1^\lambda]$.

It suffices then to show that $H_1\cap B_1^\lambda\in\mathcal{M}[H']$. Let D be $\mathbb{D}$-generic over $\mathcal{M}[y_\lambda]$ s.t. $\mathcal{M}[y_\lambda][D]=\mathcal{M}[H_0]$. Then, (by Remark 3.27) since B_1^λ is (κ^*,∞)-distributive, D is $\mathbb{D}$-generic over $\mathcal{M}[y_\lambda][H_1\cap B_1^\lambda]$. But then $\mathcal{M}[H_0][H']=\mathcal{M}[H_0][H_1\cap B_1^\lambda]$, what implies, as we have proven in Proposition 3.6, that $H_1\cap B_1^\lambda\in\mathcal{M}[H']$ i.e. $\mathcal{M}[H']=\mathcal{M}[H_1\cap B_1^\lambda]$.

Proof of the claim. By Proposition 3.3 it suffices to show that there is H s.t. $\mathcal{M}[H_0,H]$ is a generic extension of $\mathcal{M}[y_\lambda]$ via a notion of forcing T s.t. rank $T\leqslant\omega_{\kappa+22}+11$. Since then $\mathcal{M}[y_\lambda]\subseteq\mathcal{M}[H_0]\subseteq\mathcal{M}[H_0,H]$, and $\mathcal{M}[H_0]$ must be an extension of $\mathcal{M}[y_\lambda]$ via a $\mathbb{D}$ contained in r.o.(T).

Let $G_0\times G_1$ be $C(V_{\kappa+21})\times C(V_{\kappa+21})$-generic over $\mathcal{M}[H_0]$. Then there are two reals r_0, r_1 s.t. $\mathcal{M}[r_i]=\mathcal{M}[y_\lambda][G_i]$, $i=0,1$. And, by the product lemma $\mathcal{M}[y_\lambda]=\mathcal{M}[r_1]\cap\mathcal{M}[r_2]$. $\mathcal{M}\subseteq\mathcal{M}[r_i]\subseteq\mathcal{M}[H_0][G_0\times G_1]$, so by Proposition 3.3 there are $\mathbb{D}_i=(B_0*(C(V_{\kappa+21})\times C(V_{\kappa+21})))/r_i$ s.t. the biggest model is a generic extension of $\mathcal{M}[r_i]$ via $\mathbb{D}_i$, $i=0,1$. Clearly, each $\mathbb{D}_i\subseteq V_{\kappa+22}$.

Let D_i be $\mathbb{D}_i$-generic over $\mathcal{M}[r_i]$ s.t. $\mathcal{M}[H_0][G_0\times G_1]=\mathcal{M}[r_i][D_i]$.

Let H be $C(\omega_{\kappa+22})$-generic over $\mathcal{M}[H_0]$. By Theorem 3.14 there exist K_0, K_1 s.t. each K_i is $C(\omega_{\kappa+22})$-generic over $\mathcal{M}[r_i]$ and

$$\mathcal{M}[H_0][H]=\mathcal{M}[r_i][D_i][H]=\mathcal{M}[r_i][K_i],\quad i=0,1.$$

(In details: r_i plays role of y, $\mathcal{M}[r_i]$ of $\mathcal{M}$, $\mathbb{D}_i$ is $\langle C,\leqslant\rangle$, so $\mathbb{D}_i\in L_\rho[r_i]$, for some ρ, $\bar{\bar\rho}=\omega_{\kappa+22}$, i.e. $C(\rho)\simeq C(\omega_{\kappa+22})$.)

$C(\omega_{\kappa+22})$ is a homogeneous notion of forcing in $\mathcal{M}[r_i]$, so

$$(\mathrm{HOD}\ \mathcal{M}[r_i])^{\mathcal{M}[r_i][K_i]}=\mathcal{M}[r_i],\quad i=0,1.$$

Thus

$$(\mathrm{HOD}\ \mathcal{M}[r_i])^{\mathcal{M}[H_0,H]}=\mathcal{M}[r_i],\quad i=0,1.$$

Also

$$(\mathrm{HOD}\ \mathcal{M}[y_\lambda])^{\mathcal{M}[H_0,H]}\subseteq(\mathrm{HOD}\ \mathcal{M}[r_0])^{\mathcal{M}[H_0,H]}\cap(\mathrm{HOD}\ \mathcal{M}[r_1])^{\mathcal{M}[H_0,H]}.$$

Thus

$$(\mathrm{HOD}\ \mathcal{M}[y_\lambda])^{\mathcal{M}[H_0,H]}=\mathcal{M}[y_\lambda].$$

And by Theorem 3.15 there is T s.t. rank $T\leqslant\omega_{\kappa+22}+11$, and $\mathcal{M}[H_0,H]$ is a generic extension of $\mathcal{M}[y_\lambda]$ via T.

'$\subseteq$': $\mathrm{HOD}^{\mathcal{M}[y_\lambda][B_1^\lambda\cap H_1]}\subseteq(\mathrm{HOD}\ \mathcal{M}[B_1^{\lambda+1}\cap H_1])^{\mathcal{M}[y_\lambda][B_1^\lambda\cap H_1]}$. $\mathcal{M}[y_\lambda][B_1^\lambda\cap H_1]$ is an extension of $\mathcal{M}[y_\lambda][B_1^{\lambda+1}\cap H_1]$ via the homogeneous notion of forcing $S(B_1^{\lambda+1}\cap H_1)$, so

$$\mathrm{HOD}\ \mathcal{M}[B_1^{\lambda+1}\cap H_1]\subseteq(\mathrm{HOD}\ \mathcal{M}[B_1^{\lambda+1}\cap H_1])^{\mathcal{M}[y_\lambda][B^{\lambda+1}\cap H_1]}$$

(by Proposition 3.4).

Now, it remains to show that

$$(\mathrm{HOD}\ \mathcal{M}[B_1^{\lambda+1}\cap H_1])^{\mathcal{M}[y_\lambda][B_1^{\lambda+1}\cap H_1]}=\mathcal{M}[y_{\lambda+1}][B_1^{\lambda+1}\cap H_1].$$

We know that

$$\mathcal{M}[y_\lambda][B_1^{\lambda+1}\cap H_1]=\bigcap_{\alpha<\lambda}\mathcal{M}[B_0^{\alpha+1}\cap H_0][B_1^{\lambda+1}\cap H_1].$$

We have only to check that under the assumptions of the theorem the derivatives of $(B_0^{\alpha+1})_\lambda$ computed in $\mathcal{M}$, and computed in $\mathcal{M}[B_1^{\lambda+1}\cap H_1]$ are the same. This is shown in the proof of Theorem 3.29 below.

Theorem 3.29. *Let B_0, B_1 be c.b.a.'s in $\mathcal{M}\models \mathrm{ZF}+V=L$. Assume that $\bar{\bar{B}}_0\leq\kappa$, B_1 is $\kappa^*=\omega_{\omega_{\kappa+22}+12}$-closed, $\boldsymbol{B}$ is a c.b.s. of B_1, $\boldsymbol{B}\in\mathcal{M}$. Let $H_0\times H_1$ be $B_0\times B_1$-generic over $\mathcal{M}$, $\boldsymbol{H}=H_1\cap\boldsymbol{B}$. Assume that for all $\alpha<\lambda$*

$$(\mathrm{HOD}^{\alpha+1})^{\mathcal{M}[H_0]}=\mathcal{M}[H_0\cap B_0^{\alpha+1}]$$

and

$$(\mathrm{HOD}^{\alpha+1}\mathcal{M}[\boldsymbol{H}]^{\mathcal{M}[\boldsymbol{H}][H_0]}=\mathcal{M}[\boldsymbol{H}][H_0\cap B_0^{\alpha+1}].$$

Assume also that y_λ is s.t. rank $y_\lambda\leq\kappa+22$, *and*

$$\mathcal{M}[y_\lambda]=(\mathrm{HOD}^{\lambda})^{\mathcal{M}[H_0]},\qquad \mathcal{M}[y_\lambda][\boldsymbol{H}]=(\mathrm{HOD}^{\lambda}\mathcal{M}[\boldsymbol{H}])^{\mathcal{M}[\boldsymbol{H}][H_0]},$$

then

$$(\mathrm{HOD}^{\lambda+1}\mathcal{M}[\boldsymbol{H}])^{\mathcal{M}[H_0][\boldsymbol{H}]}=(\mathrm{HOD}^{\lambda+1})^{\mathcal{M}[H_0]}[\boldsymbol{H}].$$

Proof. By the results mentioned in Section 3.1. it suffices to show that the derivative of $(B_0^{\alpha+1})_\lambda$ is the same, if computed in $\mathcal{M}$ or in $\mathcal{M}[\boldsymbol{H}]$. Because then, by Theorem 3.10, the equality in the statement of Theorem 3.29 holds.

We show below the equality of the derivative of $(B_0^{\alpha+1})_\lambda$ in both models.

Definition For any $\alpha\in\mathrm{On}-\omega$, let $\aleph_0(\alpha)=\alpha$, $\aleph_{\beta+1}(\alpha)=\aleph_\beta(\alpha)^+$, $\aleph_\lambda(\alpha)=\bigcup_{\beta<\lambda}\aleph_\beta(\alpha)$.

Lemma 0. *Let H_0 be B_0-generic over $\mathcal{M}\models\mathrm{ZF}+V=L$, $\bar{\bar{B}}_0\leq\kappa$, $t\in\mathcal{M}[H_0]$,* rank $t\leq\alpha$, *$\alpha\geq\kappa$. Then $V_{\omega+\eta}^{L[t]}\subseteq L_{\aleph_\eta(\alpha)}[t]$, for any $\eta\in\mathrm{On}$.*

Proof. By induction on η. If $\eta=0$, or η-limit, then the conclusion easily follows. Assume it holds for η. We shall show that it holds for $\eta+1$.

Let $x\in L[t]$, $x\in V_{\omega+\eta+1}$, then $x\in\mathrm{Def}\,L_\beta[t]$ for some β and we can assume w.l.o.g. that $\beta\geq\aleph_\eta(\alpha)$. If $M\prec L_\beta$ and $\aleph_\eta(\alpha)\subseteq M$, and M is the smallest such model, then, as in Lemma 3.31, we can argue that $M[t]\prec L_\beta[t]$. Also $(\forall\beta'<\aleph_\eta(\alpha))$ $(L_{\beta'}[t]\subseteq M[t])$ because $\beta'\in M[t]$ and $t\in M[t]$. Therefore $x\subseteq V_{\omega+\eta}\subseteq M[t]$, by the induction hypothesis. Hence

$$y\in x\equiv L_\beta[t]\models\phi(y),\qquad y\in x\equiv M[t]\models\phi(y),\quad\text{and}\quad \pi y\in\pi x\equiv L_{\beta'}[t]\models\phi(\pi y),$$

where $\pi''M=L_{\beta'}$, and π is the collapsing map. But $\pi y=y$, $\pi t=t$, $\pi x=x$, so $y\in x\equiv L_{\beta'}[t]\models\phi(y)$. But $\bar{\bar{\beta}}'<\aleph_\eta(\alpha)^+$. Therefore $x\in L_{\beta'+1}[t]$ and $V_{\omega+\eta+1}\subseteq L_{\aleph_\eta(\alpha)^+}[t]$.

Let us consider these two models: $\mathcal{M}[\boldsymbol{H}][H_0]$ and $\mathcal{M}[H_0]$. Let

$$X_1 = \bigcap_{\alpha<\lambda} (\mathrm{HOD}^\alpha \mathcal{M}[\boldsymbol{H}])^{\mathcal{M}[\boldsymbol{H}][H_0]}, \qquad X_2 = \bigcap_{\alpha<\lambda} (\mathrm{HOD}^\alpha)^{\mathcal{M}[H_0]}.$$

By the assumption of the theorem $X_1 = \mathcal{M}[\boldsymbol{H}][y_\lambda]$ and $X_2 = \mathcal{M}[y_\lambda]$. We want to compute $(\mathrm{HOD}\ \mathcal{M}[\boldsymbol{H}])^{X_1}$ and $(\mathrm{HOD})^{X_2}$. By Theorem 3.10 in either case it suffices to compute the derivative of $(B_0^{\alpha+1})_\lambda$, but first in $\mathcal{M}[\boldsymbol{H}]$, and then in $\mathcal{M}$ (cf. Section 3.1). We shall show that they are equal.

Let $(B_0^{\alpha+1})_\lambda^*$ be the derivative in $\mathcal{M}[\boldsymbol{H}]$, and let $(B_0^{\alpha+1})_\lambda^\otimes$ be the derivative in $\mathcal{M}$.

Let $E_1(x, y)$ mean $y \in L[\boldsymbol{H}][x]$ and $E_2(x, y)$ mean $y \in L[x]$, and

$$X_i = \{y : \mathcal{M}[\boldsymbol{H}, H_0] \models E_i(y_\lambda, y)\}, \quad i = 0, 1.$$

Let t be the name of y_λ in $\mathcal{M}$. Since $\boldsymbol{B}$ is (κ^*, ∞)-distributive, t is also the name for y_λ in $\mathcal{M}[\boldsymbol{H}]$. By Proposition 3.8 and Theorem 3.10 we have:

$$(B_0^{\alpha+1})_\lambda^* = B_0^{E_1,t} \quad \text{and} \quad (B_0^{\alpha+1})_\lambda^\otimes = B_0^{E_2,t}.$$

Recall that $B_0^{E_1,t}$ is the c.b.s. of B_0 generated by $\|\varphi_{E_1}(t, \check{\alpha}, \check{F}, \check{u})\|$ where $\alpha \in \mathrm{On}$, F is a formula of ZF, and $u \in \mathcal{M}[\boldsymbol{H}]$. $B_0^{E_2,t}$ is the c.b.s. of B_0 generated by $\|\varphi_{E_2}(t, \check{\alpha}, \check{F}, \check{u})\|$ where α, F are as above and $u \in L$.

$\varphi_{E_1}(y_\lambda, \alpha, F, u)$ says $V_\alpha^{X_1} \models F(u)$,

$\varphi_{E_2}(y_\lambda, \alpha, F, u)$ says $V_\alpha^{X_2} \models F(u)$ (c.f. Section 3.1 for details).

Let $\tilde{\varphi}_{E_1}(y_\lambda, \alpha, F, u)$ say $L_\alpha[\boldsymbol{H}][y_\lambda] \models F(u)$, and $\tilde{\varphi}_{E_2}(y_\lambda, \alpha, F, u)$ say $L_\alpha[y_\lambda] \models F(u)$.

Define $\tilde{B}_0^{E_1,t}$ to be the c.b.s. of B_0 generated by $\|\tilde{\varphi}_{E_1}(t, \check{\alpha}, \check{F}, \check{u})\|$ where $u \in \mathcal{M}[\boldsymbol{H}]$ and α is a limit ordinal bigger than $\aleph_{\kappa+23}(\kappa)$. Let $\tilde{B}_0^{E_2,t}$ be the c.b.s. of B_0 generated by $\|\tilde{\varphi}_{E_2}(t, \check{\alpha}, \check{F}, \check{u})\|$, where α is a limit ordinal bigger than $\aleph_{\kappa+23}(\kappa)$ and $u \in L$. We have then four c.b.s.'s of B_0. In order to prove the theorem we need to show the following two claims:

Claim 1. $B_0^{E_1,t} = \tilde{B}_0^{E_1,t}$ and $B_0^{E_2,t} = \tilde{B}_0^{E_2,t}$.

Claim 2. $\tilde{B}_0^{E_1,t} = \tilde{B}_0^{E_2,t}$.

Having proven the claims we conclude that $B_0^{E_1,t} = B_0^{E_2,t}$, and therefore $(\mathrm{HOD}\ \mathcal{M}[\boldsymbol{H}])^{X_1} = \mathcal{M}[\boldsymbol{H}][H_0 \cap B_0^{E_1,t}]$, and $(\mathrm{HOD})^{X_2} = \mathcal{M}[H_0 \cap B_0^{E_1,t}]$, and hence

$$(\mathrm{HOD}^{\lambda+1}\mathcal{M}[\boldsymbol{H}])^{\mathcal{M}[H_0, \boldsymbol{H}]} = (\mathrm{HOD}^{\lambda+1})^{\mathcal{M}[H_0]}[\boldsymbol{H}].$$

Proof of Claim 1. We first show $B_0^{E_1,t} \subseteq \tilde{B}_0^{E_1,t}$. Let $V_\alpha^{X_1} \models F(u)$ for some $u \in \mathcal{M}[\boldsymbol{H}]$. There is cub class in On of ordinals β s.t. $L_\beta[\boldsymbol{H}][\mathrm{val}_{H_0}(t)] = V_\beta^{X_1}$, and $V_\alpha^{X_1} \models F(u) \equiv V_\beta^{X_1} \models (F(u))^{V_\alpha}$ $(X_1 = \mathcal{M}[\boldsymbol{H}][\mathrm{val}_{H_0}(t)]$ here). We pick such β. Then $\|\varphi_{E_1}(t, \check{\alpha}, \check{F}, \check{u})\| = \|\tilde{\varphi}_{E_1}(t, \check{\beta}, (\check{F})^{V_\alpha}, \check{u})\|$, because this equivalence holds for any H_0 B_0-generic over $\mathcal{M}[H_1]$.

It remains to show the opposite inclusion. Let H_0 be any B_0-generic over $\mathcal{M}[H_1]$ ultrafilter. If $\alpha > \aleph_{\kappa+23}(\kappa)$, then by Lemma 0:

$$L_\alpha[\boldsymbol{H}][V_{\kappa+23}^{L[\boldsymbol{H}][\mathrm{val}_{H_0}(t)]}] = L_\alpha[\boldsymbol{H}][\mathrm{val}_{H_0}(t)].$$

(We use also the fact that $\boldsymbol{B}$ is (κ^*, ∞)-distributive in $\mathcal{M}[\mathrm{val}_{H_0}(t)]$, also $\mathrm{rank}(\mathrm{val}_{H_0}(t)) < \kappa + 23$.) Thus for $u \in L[\boldsymbol{H}]$, $\beta \geqslant \alpha$, we have

$$L_\alpha[\boldsymbol{H}][\mathrm{val}_{H_0}(t)] \models F(u) \quad \text{iff} \quad V_\beta \models (F(u))^{L_\alpha[\boldsymbol{H}][V_{\kappa+23}]}.$$

This implies that

$$\|\tilde{\varphi}_{E_1}(t, \check{\alpha}, \check{F}, \check{u})\| = \|\varphi_{E_1}(t, \check{\beta}, (\check{F})^{L_\alpha[\boldsymbol{H}][V_{\kappa+23}]}, \check{u})\|.$$

The second equality of the claim follows by exactly the same argument.

The proof of Claim 2. In order to prove the equality $\tilde{B}_0^{E_1,t} = \tilde{B}_0^{E_2,t}$ it suffices to find $H' \in L$, and for any $\alpha, u \in L[\boldsymbol{H}]$ a pair $\alpha', u' \in L$ s.t. for any B_0-generic over $\mathcal{M}[H_1]$ set H_0, and any formula F:

$$L_\alpha[\boldsymbol{H}][\mathrm{val}_{H_0}(t)] \models F(u) \quad \text{iff} \quad L_{\alpha'}[H'][\mathrm{val}_{H_0}(t)] \models F(u').$$

Because then the value $\|\tilde{\varphi}_{E_2}(t, \check{\alpha}, \check{F}, \check{u})\|$ must be equal to $\|\tilde{\varphi}_{E_1}(t, \check{\beta}, \check{F}', \check{u}')\|$, where F' says in $L_{\beta_1}[\mathrm{val}_{H_0}(t)]$ that $F(u')$ is satisfied in $L_{\alpha'}[H'][\mathrm{val}_{H_0}(t)]$, and $\beta = \langle \beta_1, \beta_2, \alpha' \rangle$, where β_2 is the number of H' in the well-ordering of L, also β_1, α must be ordinals greater than $\aleph_{\kappa+23}(\kappa)$. This will prove $\tilde{B}_0^{E_1,t} \subseteq \tilde{B}_0^{E_2,t}$.

The opposite inclusion follows by the fact (cf. the proof of Claim 1) that $L[\mathrm{val}_{H_0}(t)]$ is a definable subclass of $L[\mathrm{val}_{H_0}(t)][\boldsymbol{H}]$ $(L[\mathrm{val}_{H_0}(t)] = L[V_{\kappa+23}])$. Thus

$$\|\tilde{\varphi}_{E_2}(t, \check{\alpha}, \check{F}, \check{u})\| = \|\varphi_{E_1}(t, \check{\alpha}, (\check{F})^{L[t]}, \check{u})\|.$$

So, it suffices to find the above H', α', u'. Let α be a fixed limit ordinal greater than $\aleph_{\kappa+23}(\kappa)$. We also fix u. W.l.o.g. we can assume that $\alpha > \mathrm{rank}\, \boldsymbol{H}$. We want also to argue that we can assume w.l.o.g. that $L_\alpha[\boldsymbol{H}]$ (and hence $L_\alpha[\boldsymbol{H}][H_0]$ and $L_\alpha[\boldsymbol{H}][y_\lambda]$) is a model of ZF^-. Because, if $L_\alpha[\boldsymbol{H}][\mathrm{val}_{H_0}(t)] \models F(u)$, then, for some $\beta > \alpha$ s.t. $L_\beta[\boldsymbol{H}][\mathrm{val}_{H_0}(t)] \models \mathrm{ZF}^-$, we have

$$L_\beta[\boldsymbol{H}][\mathrm{val}_{H_0}(t)] \models (F(u))^{L_\alpha[\boldsymbol{H}][V_{\kappa+23}]},$$

which is a formula with parameters $u, \boldsymbol{H} \in L[\boldsymbol{H}], \alpha, \kappa$.

We start with some remarks concerning the constructibility in non-transitive structures.

Let $L_\alpha[X] \models \mathrm{ZF}^-$. Then $L_\alpha[X]$ is closed under the operator Def i.e. if $y \in L_\alpha[X]$, then $\mathrm{Def}(y, \varepsilon) \in L_\alpha[X]$. Thus, if $M \prec L_\alpha[X]$, then M is also closed under Def.

Let $\mathrm{Constr}(x)$ mean 'x is constructible'. $\mathrm{Constr}(x)$ is a formula of $\mathscr{L}_{\mathrm{ZF}}$ (cf. [1] for example). $V = L \equiv (\forall x)\mathrm{Constr}(x)$ is also a formula of set theory. Also $x = L_\beta$ can be written as a formula $\mathrm{Constr}(x, \beta)$ of set theory. Thus, if $M \prec L_\alpha[X]$, M satisfies the formula $(\forall \beta)(\exists x)(x = L_\beta)$. Also $L_\beta^M = L_\beta$ (for $\beta \in M$), by the elementrary inclusion of M in $L_\alpha[X]$. L^M is naturally defined as $(\bigcup_{\alpha \in M} L_\alpha^M)^M$, and this definition can be carried out in M ($x \in L^M$ iff $(\exists z)(\exists \beta)(\mathrm{Constr}(z, \beta) \mathbin{\&} x \in z)$). Thus $L^M \prec L_\alpha$ ($L_\alpha \models \phi$ iff $L_\alpha[X] \models (\phi)^L$ iff $M \models (\phi)^L$ iff $L^M \models \phi$, where $\phi = \phi(p)$, $p \in L^M$). Therefore $(\exists \alpha' \leqslant \alpha)(L^M = L_{\alpha'})$.

Since '$x = L_\alpha[Y]$' can be written as a formula $\mathrm{Constr}(Y, x, \alpha)$ of set theory, we can define the relativized constructible hierarchy $L^M_\alpha[Y]$, $\chi \in M$:

$$L^M_{\beta+1}[Y] = \mathrm{Def}((L^M_\beta[Y], \epsilon, Y \cap L^M_\beta[Y])) \cup ((V_{\beta+1} \cap \mathrm{TC}(Y)) - V_\beta);$$

the union at the limit stages. We note that, if $\beta+1 \subseteq M$ and rank $Y \leqslant \beta$; then $L^M_{\beta'}[Y] = L_{\beta'}[Y]$ for $\beta' \leqslant \beta$. Also, if $Y \in M$, then $L^M_\beta[Y] = L_\beta[Y]$, for $\beta \in M$, because $M \prec L_\alpha[X]$. Moreover $M = L^M[M]$.

We define $M[Y]$ as $L^M[M \times \{0\} \cup Y \times \{1\}]$, but we will only consider models $M[Y]$ when rank $Y \leqslant \beta$, for some $\beta+1 \subseteq M$. Moreover Y will be given by a generic over M ultrafilter $H_0 \subseteq B_0$, and $\mathrm{TC}(\{B_0\}) \subseteq M$. Also the dense classes of B_0 will be the same as in $L_\alpha[X]$.

We note that in this case $M[H_0]$ will be the smallest ZF$^-$ model, with the same ordinals as M, s.t. M is contained in it, and H_0 belongs to it. (The standard proof works here, cf. [5], for example.)

Lemma 3.30. *If $\langle M', \{U'\}\rangle \prec \langle L_\alpha[\boldsymbol{H}], \{\boldsymbol{H}\}\rangle$, then*

(i) $U' = \boldsymbol{H}$,

(ii) *There is M s.t. $M' = M[\boldsymbol{H}]$, and $M \simeq L_{\alpha'}$), for some $\alpha' \leqslant \alpha$.*

(iii) *If $\bar{\bar{M}}' \leqslant \kappa^*$ and $M' \in L[\boldsymbol{H}]$, then $M' \simeq L_{\alpha'}[H']$, for some $H' \in L$.*

Proof. (i) $L_\alpha[\boldsymbol{H}] \vDash V = L[\boldsymbol{H}]$, so $M' \vDash V = L[U']$, but then $U' = \boldsymbol{H}$.

(ii) $M' \vDash V = L[x]$, so let $M = L^{M'}$, then $M \prec L_\alpha$ and there is $\alpha' \leqslant \alpha$ s.t. $M \simeq L_{\alpha'}$ (by the condensation lemma).

(iii) Follows by the fact that $\boldsymbol{B}$ is (κ^*, ∞)-distributive, $H' = \pi(\boldsymbol{H})$ (π is the collapsing map) must be of power less than κ^* i.e. in L.

Let $y_0 = \mathrm{val}_{H_0}(t)$, where H_0 is a generic over $\mathcal{M}[H_1]$ ultrafilter contained in B_0.

Lemma 3.31. *Let $M[\boldsymbol{H}]$ be the Skolem hull of κ^* w.r.t. $L_\alpha[\boldsymbol{H}]$ (i.e. $\kappa^* \subseteq M[\boldsymbol{H}]$). Then $M[\boldsymbol{H}][y_0] \prec L_\alpha[\boldsymbol{H}][y_0]$.*

Proof.

Claim. $M[\boldsymbol{H}][H_0] \prec L_\alpha[\boldsymbol{H}][H_0]$.

Proof of the claim. Let $L_\alpha[\boldsymbol{H}][H_0] \vDash \exists x \phi(x, p)$ with $p \in M[\boldsymbol{H}, H_0]$. By the 'truth lemma', there is $b \in H_0$ s.t. $b \Vdash \phi(\bar{x}, \bar{p})$. By the 'definability lemma', there is a formula $\mathrm{Forc}(b', \phi', \bar{x}', \bar{p})$ s.t. for any $b', \phi', \bar{x}', \bar{p}'$ the ground model satisfies it iff $b' \Vdash \phi'(\bar{x}', \bar{p}')$. So, $L_\alpha[\boldsymbol{H}] \vDash \mathrm{Forc}(b, \phi, \bar{x}, \bar{p})$ (for these particular $\phi, \bar{x}, \bar{p}, b$), i.e. $L_\alpha[\boldsymbol{H}] \vDash \exists \bar{x}\, \mathrm{Forc}(b, \phi, \bar{x}, \bar{p})$, so does $M[\boldsymbol{H}]$, but then $b \Vdash \phi(\bar{x}, \bar{p})$ for some $\bar{x} \in M[\boldsymbol{H}]$, hence $M[\boldsymbol{H}][H_0] \vDash \phi(\mathrm{val}_{H_0}(\bar{x}), p)$. This proves the claim.

Let $\bar{y}_0$ be any name for y_0, e.g. $\bar{y}_0 = t$. Let $L_\alpha[\boldsymbol{H}][y_0] \vDash \exists x \phi(x, p)$, $p \in M[\boldsymbol{H}, y_0]$, then again $b \Vdash \phi(\bar{x}, \bar{p})^{\widecheck{V[\boldsymbol{H}][\bar{y}_0]}}$, for some $b \in H_0$, hence

$$L_\alpha[\boldsymbol{H}] \vDash \exists \bar{x}\, \mathrm{Forc}(b, \phi^{\widecheck{V[\boldsymbol{H}][\bar{y}_0]}}, \bar{x}, \bar{p}),$$

and

$$M[\boldsymbol{H}] \vDash \exists \bar{x}_0\, \mathrm{Forc}(b, \phi^{\widecheck{V[\boldsymbol{H}][\bar{y}_0]}}, \bar{x}_0, \bar{p}),$$

i.e.

$$M[\boldsymbol{H}][H_0]\models(\phi(\mathrm{val}_{H_0}(x_0), p))^{V[\boldsymbol{H}][y_0]},$$

i.e.

$$M[\boldsymbol{H}][y_0]\models\phi(x_0, p).$$

This completes the proof of Lemma 3.31.

So now, if $L_\alpha[y_0][\boldsymbol{H}]\models F(u)$, we have that $L_{\alpha'}[y_0][H']\models F(u')$, where $u'\in L$, by the same argument which shows that $H'\in L$, (u' is the collapse of u).

Since this holds for *any* H_0 B_0-generic over $\mathcal{M}[H_1]$, we conclude that:

$$\|L_\alpha[\check{y}_0][\check{\boldsymbol{H}}]\models F(\check{u})\| = \|L_{\alpha'}[\check{y}_0][\check{H}']\models F(\check{u}')\|.$$

So finally Claim 2 and Theorem 3.29 are proven.

4. Ordinal definability II

In this part we prove the theorem which describes the structure of iterated HOD when the universe is obtained by product of two generic sets. We use here the results from the previous section. The second part of the chapter generalizes the obtained results to the case when one forces with a proper class.

4.0

Theorem 4.1. *Let B_0, B_1 be c.b.a.'s in $\mathcal{M}\models\mathrm{ZFC}+V=L$, $\bar{\bar{B}}_0\leqslant\kappa$. Let B_1 be $\kappa^*=\omega_{\omega_{\kappa+22}+12}$-closed. Let $H_0\times H_1$ be $B_0\times B_1$-generic over $\mathcal{M}$, and $\alpha\in\mathrm{On}$. Then there exists $y_\alpha\in\mathcal{M}[H_0, H_1]$ s.t.*

(1) $(\mathrm{HOD}^\alpha\mathcal{M}[H_1\cap\boldsymbol{B}])^{\mathcal{M}[H_0,H_1\cap\boldsymbol{B}]}=\mathcal{M}[y_\alpha][H_1\cap\boldsymbol{B}]$, *for any c.b.s. $\boldsymbol{B}$ of B_1.*

(2) *For every $\bar{\rho}\geqslant\omega_{\kappa+22}$ s.t. $L_{\bar{\rho}}\models\mathrm{ZF}$, $L_{\bar{\rho}}[y_\alpha]=(\mathrm{HOD}^\alpha)^{L_{\bar{\rho}}[H_0]}$.*

(3) $\mathcal{M}[y_\alpha]=(\mathrm{HOD}^\alpha\mathcal{M})^{\mathcal{M}[H_0]}$.

Proof. Since (3) follows by (1) ($\boldsymbol{B}=\{\mathbb{0}, \mathbb{1}\}$), it suffices to show the first two points. Assume that the theorem holds for α's less than λ. We shall show it holds for λ.

We shall find y_λ s.t. $(\mathrm{HOD}^\lambda\boldsymbol{\mathcal{M}})^{\boldsymbol{\mathcal{M}}[H_0]}=\boldsymbol{\mathcal{M}}[y_\lambda]$, where $\boldsymbol{\mathcal{M}}$ denotes either $L_{\bar{\rho}}$ or $\mathcal{M}[H_1\cap\boldsymbol{B}]$. That will prove the theorem.

Lemma 4.2. *Let H be $C(\kappa)$-generic over $\boldsymbol{\mathcal{M}}[H_0]$. Then $(\mathrm{HOD}\,\boldsymbol{\mathcal{M}}[y])^{\boldsymbol{\mathcal{M}}[H_0,H]}=\boldsymbol{\mathcal{M}}[y]$, for any y A-generic over $\boldsymbol{\mathcal{M}}$, where A is a c.b.s. of B_0.*

Proof. By Theorem 3.14 ($\bar{\bar{\rho}}=\kappa, p>\kappa, C=S(H_0\cap A)$), $\boldsymbol{\mathcal{M}}[y][H_0][H]=\boldsymbol{\mathcal{M}}[y][K]$, where K is $C(\kappa)$-generic set over $\boldsymbol{\mathcal{M}}[y]$. Since $C(\kappa)$ is homogeneous, by Theorem 3.5, $(\mathrm{HOD}\,\boldsymbol{\mathcal{M}}[y])^{\boldsymbol{\mathcal{M}}[y][H_0][H]}=\boldsymbol{\mathcal{M}}[y]$.

Lemma 4.3. *Let H be as above.*

(A) *If*

$$X = (\mathrm{HOD}^\lambda \mathcal{M}[\boldsymbol{B} \cap H_1])^{\mathcal{M}[H_0, \boldsymbol{B} \cap H_1]},$$

then

$$X = (\mathrm{HOD}(\mathcal{M}[\boldsymbol{B} \cap H_1] \cup X))^{\mathcal{M}[H_0, \boldsymbol{B} \cap H_1, H]}.$$

(B) *If*

$$X = (\mathrm{HOD}^\lambda)^{L_{\bar\rho}[H_0]},$$

then

$$X = (\mathrm{HOD}(L_{\bar\rho} \cup X))^{L_{\bar\rho}[H_0, H]}.$$

Proof. By Theorem 3.10 for all $\alpha < \lambda$

$$\mathrm{HOD}^{\alpha+1}\mathcal{M}[\boldsymbol{B} \cap H_1] = \mathcal{M}[\boldsymbol{B} \cap H_1][B_0^{\alpha+1} \cap H_0] \quad \text{and}$$
$$(\mathrm{HOD}^{\alpha+1})^{L_{\bar\rho}[H_0]} = L_{\bar\rho}[B_0^{\alpha+1} \cap H_0].$$

Note that the induction hypothesis is used here to make sure that $B_0^{\alpha+1}$ is in both cases the same.

Let X be given by (A) or (B), $\mathcal{M}$ be an appropriate model. Then $\mathcal{M} \cup X \subseteq \mathcal{M}[H_0 \cap B_0^{\alpha+1}]$, for all $\alpha < \lambda$. Therefore

$$(\mathrm{HOD}\, \mathcal{M} \cup X)^{\mathcal{M}[H_0, H]} \subseteq \bigcap_{\alpha<\lambda} (\mathrm{HOD}\, \mathcal{M}[H_0 \cap B_0^{\alpha+1}])^{\mathcal{M}[H_0, H]}.$$

Lemma 4.2 (with $y = B_0^{\alpha+1} \cap H_0$) completes the proof.

Now we apply Theorem 3.26. Let $B = B_0 \times C(\kappa)$. Then $H_0 \times H$ is B-generic over $\mathcal{M}$. If X is given by (A) or (B) of Lemma 4.3, then, by the last lemma, $X = \mathrm{HOD}\, \mathcal{M} \cup X$. X is a model of ZF because, by the induction hypothesis, the sequence $\mathrm{HOD}^\alpha \mathcal{M}$, $\alpha < \lambda$, is definable. By the proof of Theorem 3.26 $X = \mathrm{HOD}\, \mathcal{M}[X \cap V_{\kappa+11}]$, but

$$X \cap V_{\kappa+11} = V_{\kappa+11} \cap \bigcap \{L[B_0^{\alpha+1} \cap H_0] : \alpha < \lambda\}$$

is the same in both case ($\bar\rho > \omega_{\kappa+22}$, $\mathcal{M} \cap V_{\kappa+23} \subseteq L$). Also $X = \mathcal{M}[T]$, and $T \in L_{\bar\rho}[H_0]$ because $\bar\rho > \omega_{\kappa+22}$, and rank $T < \kappa + 22$. So, putting $y_\lambda = T$, we have satisfied both (1) and (2), moreover the sequence $\mathrm{HOD}^\alpha \mathcal{M}[H_1 \cap \boldsymbol{B}]$, $\alpha \leqslant \lambda$, is uniformly definable, as the sequence $(\mathrm{HOD}^\alpha)^{L_{\bar\rho}[H_0]}$ is uniformly definable in $\mathcal{M}[H_0, H_1 \cap \boldsymbol{B}]$.

This proves the limit case of the theorem.

The step from $\alpha+1$ to $\alpha+2$ follows immediately by Theorem 3.5 and the fact that $\boldsymbol{H}$ cannot add any new automorphism of $B_0^{\alpha+1}$, (cf. Remark 3.27). The step from λ to $\lambda+1$ follows by Theorem 3.29 and Theorem 3.28, and the present theorem proven for α's less than λ.

4.1

Let P_i, $i<\mathrm{On}$, be notions of forcing in $\mathcal{M}$. Let

$$\mathbb{P}=\prod_{i<\mathrm{On}} P_i=\{f\colon(\exists\gamma)(\mathrm{dom}\, f\subseteq\gamma\ \&\ (\forall\alpha)(\alpha\in\mathrm{dom}\, f\Rightarrow f(\alpha)\in P_\alpha))\}.$$

As in Section 2.1 we define $\mathbb{P}_\alpha$ and $\mathbb{P}^\alpha$.

Assume that, for each α, $\bar{\bar{\mathbb{P}}}_\alpha\leq\kappa_\alpha$ and $\mathbb{P}^\alpha$ is κ_α-closed, where $\langle\kappa_\alpha\colon\alpha\in\mathrm{On}\rangle$ is an ascending sequence of cardinals. Let $\mathbb{P}$ be κ_0-closed, let $B=\mathrm{r.o.}(\mathbb{P})$.

We are going to define a sequence B^α, $\alpha<\kappa_0^+$, of c.b.s.'s of B s.t. for any B-generic over $\mathcal{M}$ ultrafilter G, we have $(\mathrm{HOD}^\alpha\mathcal{M})^{\mathcal{M}[G]}=\mathcal{M}[G\cap B^\alpha]$.

Definition 4.1. Algebras A and A', which are c.b.s.'s of B, are said to be *locally equal*, if for any G B-generic over $\mathcal{M}$, $\mathcal{M}[A\cap G]=\mathcal{M}[A'\cap G]$. We denote this as $A=_1A'$.

Thus $B=_1(B)_\eta\times(B)^\eta$, where $(B)_\eta=\mathrm{r.o.}(\mathbb{P}_\eta)$, and $(B)^\eta=\mathrm{r.o.}(\mathbb{P}^\eta)$. Also $B=_1\bigcup_{\eta\in\mathrm{On}}(B)_\eta$. Moreover, since each $\mathbb{P}^\alpha$ can be decomposed in the same way as $\mathbb{P}$ into $(\mathbb{P}^\alpha)_\eta\times(\mathbb{P}^\alpha)^\eta$, $\eta\geq\alpha$,

$$\mathrm{r.o.}(\mathbb{P}^\alpha)=_1\bigcup_{\eta\in\mathrm{On}-\alpha}\mathrm{r.o.}((\mathbb{P}^\alpha)_\eta).$$

Definition 4.2. If A is a c.b.s. of B, and $A=_1\bigcup_{\eta\in\mathrm{On}}(A)_\eta$, where $(A)_\eta$'s are c.b.s.'s of B, then we define $A^*=\bigcup_{\eta\in\mathrm{On}}((A)_\eta)^*$.

Proposition 4.4. *Assume B_0, B_1 are c.b.a.'s in $\mathcal{M}\models\mathrm{ZF}$, $\bar{\bar{B}}_0\leq\kappa$, B_1 is a class in $\mathcal{M}$, $B_0\Vdash$'$\check{B}_1$ is $(\check{\kappa},\infty)$-distributive'. Assume that for all $\eta\in\mathrm{On}$,*

$$B_1=_1(B_1)_\eta\times(B_1)^\eta\quad\text{and}\quad(\bar{\bar{B}}_1)_\eta\leq\kappa_\eta,$$

and $B_0\times(B_1)_\eta\Vdash$'$(\check{B}_1)^\eta$ is $(\check{\kappa}_\eta,\infty)$-distributive', where $\langle\kappa_\eta\colon\eta\in\mathrm{On}\rangle$ is an ascending sequence of cardinals.

Let $H_0\times H_1$ be $B_0\times B_1$-generic over $\mathcal{M}$. Let $\mathcal{M}[H_0,H_1]\models\mathrm{ZF}$. Then

$$(\mathrm{HOD}\,\mathcal{M})^{\mathcal{M}[H_0,H_1]}=\mathcal{M}[H_0\cap B_0^*][H_1\cap B_1^*].$$

Remark 4.5. This is nothing but a version of Proposition 3.6 for the case when B_1 is a class.

Proof. As in the proof of Proposition 3.6 we can argue that $H_0\cap B_0^*$, $H_1\cap(B_1)_\eta^*$ are definable for all $\eta\in\mathrm{On}$. This gives '$\supseteq$'.

On the other hand $B_0\times B_1$ can be decomposed, for all $\eta\in\mathrm{On}$, into $(B_0\times(B_1)_\eta)\times(B_1)^\eta$. By the proof of Proposition 3.7, and using the fact that $(B_0\times(B_1)_\eta)^*=_1B_0^*\times((B_1)_\eta)^*$, (cf. Proposition 3.6), we conclude '$\subseteq$'.

Let $B^0=B$, and assume we have defined B^α. We are going to define $B^{\alpha+1}$.

It may be useful to introduce the following convention: The expression $A =_1 (A)_\eta \times (A)^\eta$ should be read: 'there are $(A)_\eta$ and $(A)^\eta$ s.t. $A =_1 (A)_\eta \times (A)^\eta$, and $(\bar{A})_\eta \leq \kappa_\eta$, and $(A)_\eta \Vdash$'$(\check{A})_\eta$ is $(\check{\kappa}_\eta, \infty)$-distributive', where $\langle \kappa_\eta : \eta \in \mathrm{On}\rangle$ is the sequence of cardinals given for B.

Assume that B^α satisfies the following induction hypotheses:

(1) For any $\eta \in \mathrm{On}$, $B^\alpha =_1 (B^\alpha)_\eta \times (B^\alpha)^\eta$.

(2) For any $\eta \in \mathrm{On}$, any $\alpha' < \alpha$, $(B^\alpha)_\eta$ is a c.b.s. of both B^α and $(B^{\alpha'})_\eta$.

(3) For any $\eta \in \mathrm{On}$, any $\eta' \geq \eta$, $(B^\alpha)^\eta =_1 ((B^\alpha)^\eta)_{\eta'} \times ((B^\alpha)^\eta)^{\eta'}$ and $((B^\alpha)^\eta)^{\eta'} = (B^\alpha)^{\eta'}$, and $((B^\alpha)^\eta)_{\eta'}$ is a c.b.s. of $(B^\alpha)^\eta$ and of $((B^{\alpha'})^\eta)_{\eta'}$, for any $\alpha' < \alpha$.

(4) $(B^\alpha)_{\eta'} =_1 (B^\alpha)_\eta \times ((B^\alpha)^\eta)_{\eta'}$, for any η, η', $\eta' \geq \eta$.

(5) $B^\alpha =_1 \bigcup_{\eta < \mathrm{On}} (B^\alpha)_\eta$, $(B^\alpha)^\eta =_1 \bigcup_{\eta' \geq \eta} ((B^\alpha)^\eta)_{\eta'}$, $\eta \in \mathrm{On}$.

Remark 4.6. One can easily check that B^0 satisfies (1)–(5).

Assume also that $[\mathrm{HOD}^\alpha \mathcal{M}]^{\mathcal{M}[G]} = \mathcal{M}[G \cap B^\alpha]$, for an G B-generic over $\mathcal{M}$, and that

$$(\mathrm{HOD}^\alpha \mathcal{M})^{\mathcal{M}[G \cap (B)^\eta]} = \mathcal{M}[(B^\alpha)^\eta \cap G].$$

From now on, let G be any generic ultrafilter on B.

We define:

(i) $\quad B^{\alpha+1} =$ the c.b.s. of B generated by $(B^\alpha)^*$.

(ii) $\quad (B^{\alpha+1})^\eta =$ the c.b.s. of B generated by $((B^\alpha)^\eta)^*$, $\quad \eta \in \mathrm{On}$.

(iii) $\quad (B^{\alpha+1})_\eta = ((B^\alpha)_\eta)^*$, $\quad \eta \in \mathrm{On}$.

(iv) $\quad ((B^{\alpha+1})^\eta)_{\eta'} = (((B^\alpha)^\eta)_{\eta'})^*$, $\quad \eta, \eta' \in \mathrm{On}$, $\eta' \geq \eta$.

(v) $\quad ((B^{\alpha+1})^\eta)^{\eta'} = (B^{\alpha+1})^{\eta'}$, $\quad \eta, \eta' \in \mathrm{On}$, $\eta' \geq \eta$.

We first check that $(\mathrm{HOD}^{\alpha+1} \mathcal{M})^{\mathcal{M}[G]} = \mathcal{M}[G \cap B^{\alpha+1}]$. But this follows by the proof of Proposition 3.7.

Now, we have to check that the conditions (1)–(5) are satisfied by $B^{\alpha+1}$. (5) follows by the proof of Proposition 3.7.

By Proposition 4.4 we have that

$$(\mathrm{HOD}^{\alpha+1} \mathcal{M})^{\mathcal{M}[G]} = \mathcal{M}[((B^\alpha)_\eta)^* \cap G][((B^\alpha)^\eta)^* \cap G],$$

thus $B^{\alpha+1} =_1 (B^{\alpha+1})_\eta \times (B^{\alpha+1})^\eta$.

$(B^{\alpha+1})_\eta \Vdash$'$(\check{B}^{\alpha+1})^\eta$ is (κ_η, ∞)-distributive', because by the induction hypothesis, $(B^\alpha)^\eta$ is forced by $(B^\alpha)_\eta$ to be (κ_η, ∞)-distributive, so $(B^\alpha)_\eta$ forces $(B^{\alpha+1})^\eta$ (which is a c.b.s. of $(B^\alpha)^\eta$) to be (κ_η, ∞)-distributive. Thus so must force $(B^{\alpha+1})_\eta$ which is a c.b.s. of $(B^\alpha)_\eta$. (This follows by Remark 3.27.) Hence (1) holds.

(2) follows since $(B^{\alpha+1})_\eta$ is contained in the class of generators of $B^{\alpha+1}$.

(3) holds: $(B^{\alpha+1})_\eta =_1 ((B^{\alpha+1})^\eta)_{\eta'} \times ((B^{\alpha+1})^\eta)^{\eta'}$ by Proposition 4.4 and the induction hypotheses, $((B^{\alpha+1})^\eta)_{\eta'}$ is contained in the class of generators of $(B^{\alpha+1})^\eta$, A^* is always a c.b.s of a c.b.a. A.

(4) holds: this follows by Proposition 3.6 and the induction hypothesis.

The required distributivity of the upper parts of the algebras in the cases (3) and (4) follows exactly as in the proof of (1). So, we have proven that, if B^{α} is given and satisfies (1)–(5), then $B^{\alpha+1}$ can be defined and satisfies (1)–(5).

We are left with the *limit case.* Assume that for all $\alpha<\lambda$, we have defined B^{α} s.t. B^{α} satisfies (1)–(5), and $\mathrm{HOD}^{\alpha}\mathcal{M}=\mathcal{M}[G\cap B^{\alpha}]$.

We define:

(i) $$(B^{\lambda})_{\eta}=\bigcap_{\alpha<\lambda}(B^{\alpha})_{\eta},\quad \eta\in\mathrm{On}.$$

(ii) $$((B^{\lambda})^{\eta})_{\eta'}=\bigcap_{\alpha<\lambda}((B^{\alpha})^{\eta})_{\eta'},\qquad \eta,\eta'\in\mathrm{On},\quad \eta'\geqslant\eta.$$

(iii) $$B^{\lambda}=\text{the c.b.s. of } B \text{ generated by } \bigcup_{\eta<\mathrm{On}}(B^{\lambda})_{\eta}.$$

(iv) $$(B^{\lambda})^{\eta}=\text{the c.b.s. of } B \text{ generated by } \bigcup_{\eta\leqslant\eta'<\mathrm{On}}((B^{\lambda})^{\eta}).$$

(v) $$((B^{\lambda})^{\eta})^{\eta'}=(B^{\lambda})^{\eta'},\quad \eta'\geqslant\eta.$$

We have to show that $(\mathrm{HOD}^{\lambda}\mathcal{M})^{\mathcal{M}[G]}=\mathcal{M}[G\cap B^{\lambda}]$, and that (1)–(5) are satisfied.

We first show that $\bigcap_{\alpha<\lambda}\mathcal{M}[G\cap B^{\alpha}]=\mathcal{M}[G\cap B^{\lambda}]$: We note that '$\supseteq$' holds because B^{λ} is a c.b.s. of each B^{α}, that implies the inclusion of models.

'$\subseteq$' is true since, if x is in the intersection, then there is η s.t., for every $\alpha<\lambda$, $x\in\mathcal{M}[G\cap(B^{\alpha})_{\eta}]$. But then $x\in\mathcal{M}[G\cap(B^{\lambda})_{\eta}]$ (by Proposition 1.7). But $(B^{\lambda})_{\eta}$ is a c.b.s of B^{λ}, so the inclusion follows.

$(\mathrm{HOD}^{\lambda}\mathcal{M})^{\mathcal{M}[G\cap(B)^{\eta}]}=\mathcal{M}[(B^{\lambda})^{\eta}\cap G]$, can be proven in the same manner.

Thus we have also proven (5). (2) is obvious.

We prove (1): we must show that

$$\mathcal{M}[G\cap(B^{\lambda})_{\eta}][G\cap(B^{\lambda})^{\eta}]=\mathcal{M}[G\cap B^{\lambda}].$$

'$\supseteq$': It suffices to show that, for each $\eta'\geqslant\eta$, $g=((B^{\lambda})^{\eta})_{\eta'}\cap G$ is in $\mathcal{M}[G\cap B^{\lambda}]$. But

$$g=\bigcap_{\alpha'<\alpha<\lambda}(((B^{\alpha})^{\eta})_{\eta'}\cap(G\cap((B^{\alpha'})^{\eta})_{\eta'}))$$

is in every $\mathcal{M}[G\cap B^{\alpha}]$, so must be in their intersection.

'$\subseteq$': We note that $\mathcal{M}[B^{\lambda}\cap G]\subseteq\mathcal{M}[(B^{\alpha})_{\eta}\cap G][(B^{\alpha})^{\eta}\times G\}$, for all $\alpha<\lambda$. Thus it is contained in $\bigcap_{\alpha<\lambda}\mathcal{M}[(B^{\alpha})_{\eta}\cap G][(B^{\alpha_0})^{\eta}\cap G]$, for each $\alpha_0<\lambda$. So (by Proposition 1.7), it is contained in $\mathcal{M}[(B^{\lambda})_{\eta}\cap G][(B^{\alpha_0})^{\eta}\cap G]$, for each $\alpha_0<\lambda$. Hence, if $x\in\mathcal{M}[B^{\lambda}\cap G]$, then there is $\eta'\geqslant\eta$ s.t.

$$x\in\mathcal{M}[(B^{\lambda})_{\eta}\cap G][((B^{\alpha_0})^{\eta})_{\eta'}\cap G],\quad \text{for all } \alpha_0<\lambda.$$

Thus $x\in\mathcal{M}[(B^{\lambda})_{\eta}\cap G][((B^{\lambda})^{\eta})_{\eta'}\cap G]$, and the last model is contained in $\mathcal{M}[G\cap(B^{\lambda})_{\eta}\times(B^{\lambda})^{\eta}]$, because $((B^{\lambda})^{\eta})_{\eta'}$ is a c.b.s. of $(B^{\lambda})^{\eta}$. This proves (1).

(3) can be proven exactly as (1), just $(B^{\lambda})^{\eta}$ plays the role of B^{λ}. Also, by exactly the same argument, we show (4). The required distributivity of the upper parts of the c.b.a.'s follows as in the proof of (1) for $\alpha+1$.

Proposition 4.7. *Assume all the hypotheses of Theorem* 3.28, *except that now* $B_1 = B$ (*i.e is a class of conditions*). *Then the conclusion of Theorem* 3.28 *holds.*

Proposition 4.8. *Assume all the hypotheses of Theorem* 3.29, *except that now* $B_1 = B$ (*i.e. is a class of conditions*). *Then the conclusion of Theorem* 3.29 *holds.*

Proof of Proposition 4.7. Let $y_{\lambda+1}$ be s.t. $(\mathrm{HOD}^{\lambda+1})^{\mathcal{M}[H_0]} = \mathcal{M}[y_{\lambda+1}]$. Then as before we have to show the two inclusions.

'$\supseteq$': This can be prove.. exactly as before. We prove that $\mathcal{M}[y_\lambda]$, and each of $\mathcal{M}[H_1 \cap (B_1^\lambda)_\eta]$ are definable classes in HOD^λ. And then $H_1 \cap ((B_1^\lambda)_\eta)^*$ is definable for every η.

'$\subseteq$': We want to show that

$$(\mathrm{HOD}\,\mathcal{M}[B_1^{\lambda+1} \cap H_1])^{\mathcal{M}[y_\lambda][B_1^\lambda \cap H_1]} \subseteq (\mathrm{HOD}\,\mathcal{M}[B_1^{\lambda+1} \cap H_1])^{\mathcal{M}[y_\lambda][B_1^{\lambda+1} \cap H_1]}.$$

So, let x belong to the left-hand side, then there is η s.t. $x \in (\mathrm{HOD}\,\mathcal{M}[(B_1^{\lambda+1})_\eta \cap H_1]^{\mathcal{M}[y_\lambda][(B_1^\lambda)_\eta \cap H_1]}$, (by the reflection principle). As before, we can argue that $\mathcal{M}[y_\lambda][(B_1^\lambda)_\eta \cap H_1]$ is an extension of $\mathcal{M}[y_\lambda][(B_1^{\lambda+1})_\eta \cap H_1]$ via a homogeneous notion of forcing, and thus (by Proposition 3.4)

$$x \in (\mathrm{HOD}\,\mathcal{M}[(B_1^{\lambda+1})_\eta \cap H_1])^{\mathcal{M}[y_\lambda][(B_1^{\lambda+1})_\eta \cap H_1]}.$$

The inclusion follows by the fact that $\mathcal{M}[y_\lambda][(B_1^{\lambda+1})_\eta \cap H_1]$ is a definable subclass of $\mathcal{M}[y_\lambda][(B_1^{\lambda+1} \cap H_1]$, (cf. the proof of '$\supseteq$').

The inclusion '$\subseteq$' is then a consequence of the equality of the derivatives of $(B_0^{\alpha+1})_\lambda$, in the present case — the consequence of Proposition 4.8.

Proof of Proposition 4.8. This goes exactly as before, because the only thing to do is to compute the derivatives of $(B_0^\alpha)_\lambda$, but this derivative is generated by a set of values, so there is η s.t. all the elements used in the computation are in $\mathcal{M}[\boldsymbol{H} \cap (\boldsymbol{B})_\eta]$, then the equality of the derivatives follows by Theorem 3.29 $(\boldsymbol{B})_\eta \overset{\mathrm{df}}{=} (B)_\eta \cap \boldsymbol{B})$.

Proposition 4.9. *Under the hypotheses of Theorem* 4.1, *except that* $B_1 = B$ (*i.e. is a class of conditions*), $\boldsymbol{B}$ *is a subclass of* B_1, *the conclusions* (1), (2) *and* (3) *of Theorem* 4.1 *hold.*

Proof. As before, using Propositions 4.7 and 4.8 in place of Theorem 3.28 and Theorem 3.29.

Proposition 4.10. *Under the hypotheses of Proposition* 3.13, *except that now* $B_1 = B$, *the conclusion of Proposition* 3.13 *holds.*

Proof. The proof of '$\supseteq$' is as before. '$\subseteq$' follows as before, except that we are now using the argument which showed that $\bigcap_{\alpha<\lambda} \mathrm{HOD}^\alpha \mathcal{M} = \mathcal{M}[G \cap B^\lambda]$ (in the proof of existence of B^α's) to prove that

$$\bigcap_{\beta<\lambda} \mathcal{M}[B_0^{\alpha_0+1} \cap H_0][H_1 \cap B_1^{\beta+1}] = \mathcal{M}[B_0^{\alpha_0+1} \cap H_0][H_1 \cap B_1^\lambda].$$

5. The result

Let M, C, Γ_δ be as in Section 2.

Definition 5.1. $B^0 = \mathrm{r.o.}(C)$, $B_\delta^0 = \mathrm{r.o.}(\prod_{\delta'>\delta} T_{\gamma'}^0)$, $B^0 \restriction \delta = \mathrm{r.o.}(\prod_{\delta'<\delta} T_{\delta'}^0)$. The sequences $B^\alpha \restriction \delta = (B^0 \restriction \delta)^\alpha$, $B_\delta^\alpha = (B_\delta^0)^\alpha$ are defined as in Section 4.1.

Theorem 5.1. *Let G be B^0-generic over M. Then, for each $\alpha \leqslant \Gamma_\delta$,*

$$(\mathrm{HOD}^{\alpha+1})^{M[G]} = M[G \cap B^{\alpha+1} \restriction \delta][b_\delta^{\alpha+1}][G \cap B_\delta^{\alpha+1}].$$

Proof. By induction on α. Assume the theorem holds for α, we will show it holds for $\alpha+1$.

We apply Proposition 3.6 and Proposition 4.4 and obtain:

$$\mathrm{HOD}^{\alpha+2} = M[G \cap (B^{\alpha+1} \restriction \delta)^*][b_\delta^{\alpha+1} \cap (\mathrm{r.o.}(T_\delta^{\alpha+1})^*][G \cap (B_\delta^{\alpha+1})^*].$$

But

$$(\mathrm{r.o.}(T_\delta^{\alpha+1}))^* = (\mathrm{r.o.}(T_\delta^{\alpha+2}))^* \quad \text{and} \quad b_\delta^{\alpha+1} \cap \mathrm{r.o.}(T_\delta^{\alpha+2}) = b_\delta^{\alpha+2},$$

by Theorem 2.1 and Proposition 2.2.

Assume the theorem holds for all $\alpha < \lambda$. We show below that it holds for λ. By Proposition 3.13 and Proposition 4.10

$$\mathrm{HOD}^\lambda = \bigcap_{\alpha<\lambda} M[B^{\alpha+1} \restriction \delta+1 \cap G][B_{\delta+1}^\lambda \cap G].$$

By Proposition 4.9 we conclude that

$$\begin{aligned} \mathrm{HOD}^\lambda &= \bigcap_{\alpha<\lambda} M[B_{\delta+1}^\lambda \cap G][B^{\alpha+1} \restriction \delta+1 \cap G] \\ &= \bigcap_{\alpha<\lambda} (\mathrm{HOD}^\alpha M[B_{\delta+1}^\lambda \cap G])^{M[B^0 \restriction \delta+1 \cap G, B_{\delta+1}^\lambda \cap G]} \\ &= (\mathrm{HOD}^\lambda M[B_{\delta+1}^\lambda \cap G])^{M[B^0 \restriction \delta+1 \cap G, B_{\delta+1}^\lambda \cap G]} \\ &= M[y_\lambda][B_{\delta+1}^\lambda \cap G], \quad \text{where } M[y_\lambda] = (\mathrm{HOD}^\lambda)^{M[B^0 \restriction \delta+1]}. \end{aligned}$$

Hence, by Theorem 3.28 and Proposition 4.7, we have:

$$(\mathrm{HOD}^{\lambda+1})^{M[G]} = M[B^{\lambda+1} \restriction \delta+1 \cap G][B_\delta^{\lambda+1} \cap G].$$

Since $\lambda \leqslant \Gamma_\delta$, and T_δ^0 is κ-closed, where $\kappa^{++} = \Gamma_\delta^+$, we have (by exactly the same arguments applied to $B^0 \restriction \delta+1$) that $B^{\lambda+1} \restriction \delta+1 = \mathrm{r.o.}(B^{\lambda+1} \restriction \delta \times T_\delta^{\lambda+1})$. Therefore

$$(\mathrm{HOD}^{\lambda+1})^{M[G]} = M[B^{\lambda+1} \restriction \delta \cap G][b_\delta^{\lambda+1}][B_\delta^{\lambda+1} \cap G].$$

Corollary 5.2. *The sequence* HOD$^\alpha$: $\alpha <$ On *is definable in* $M[G]$.

Proof. For any β we can find $\Gamma_\delta \geq \beta$, and if $\alpha < \beta$, then

$$\mathrm{HOD}^{\alpha+1} = M[G \cap B^{\alpha+1} \restriction \delta][b_\delta^{\alpha+1}][G \cap B_\delta^{\alpha+1}].$$

$T_\delta^{\alpha+1}$ is definable in M, also $B^{\alpha+1} \restriction \delta$ is definable in M (by Theorem 3.10), the same applies to $B_\delta^{\alpha+1}$ (cf. Section 4.1).

Corollary 5.3. *The sequence* HOD$^\alpha$: $\alpha <$ On, *is a strictly descending sequence of models for* ZF.

Proof. HOD$^\alpha \models$ ZF, by Corollary 5.2. The sequence is descending because $b_\delta^{\alpha+1} \notin$ HOD$^{\alpha+2}$, $\alpha < \Gamma_\delta$, $\delta \in$ On. This can be shown by the following argument: r.o.$(T_\delta^{\alpha+1})$ and r.o.$(T_\delta^{\alpha+2})$ are (Γ_δ, ∞)-distributive, thus $G \cap B^0 \restriction \delta$ is $B^0 \restriction \delta$-generic over both $M[b_\delta^{\alpha+2}]$ and $M[b_\delta^{\alpha+1}]$ $(\overline{B^0 \restriction \delta} < \Gamma_\delta)$.

If $b_\delta^{\alpha+1} \in M[b_\delta^{\alpha+2}][B^0 \restriction \delta \cap G]$, then (by Proposition 1.3) we have a contradiction with $M[b_\delta^{\alpha+1}] \neq M[b_\delta^{\alpha+2}]$ (cf. Section 2.0). Hence

$$b^{\alpha+1} \notin \mathrm{HOD}^{\alpha+2} \subseteq M[b_\delta^{\alpha+2}][B^0 \restriction \delta \cap G][B_\delta^0 \cap G].$$

Theorem 5.4 (The main result). Con(ZF) *implies*

Con(ZF + 'HOD$^\alpha$: $\alpha <$ On, *is strictly descending*')

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